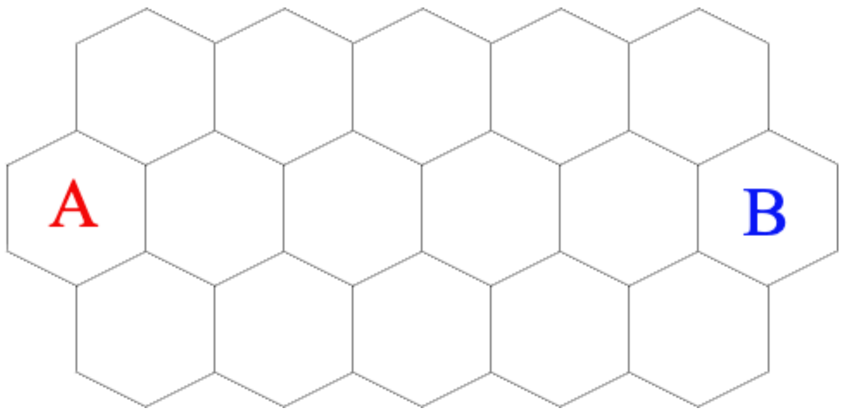
1. Matrix Tessellation Laplace writes the decimal number 2024 in base 3 on a blackboard. He writes another decimal number $0 \leq N \leq 2023$ in base 3 underneath and subtracts the bottom number from the top number. He notices that as he performs the subtraction from right to left, at no point does he need to borrow (regroup). Compute the number of possible values of N.

   Solution: Note that $2024 = 2202222_3$. Therefore, the last four digits of N can be anything (0, 1, or 2). The 5th digit from the right must be 0, and the last two digits once again can be anything. All such resulting N constructed this way are between 0 and 2024, inclusive, since the digit in each place of N is at most that of 2024. As there are six 2's in 2202222_3, the answer is $3^6 - 1 = \boxed{728}$, since N cannot be 2024.

2. How many ways are there to color each hexagon below either red (denoted by A) or blue (denoted by B) such that the region of all red hexagons is contiguous, the region of all blue hexagons is contiguous, and each hexagon borders at least two hexagons of the same color? The leftmost and rightmost hexagons are already colored.



   Solution: Let r be the number of red hexagons. Doing casework on r, we see manually inspect and see that...

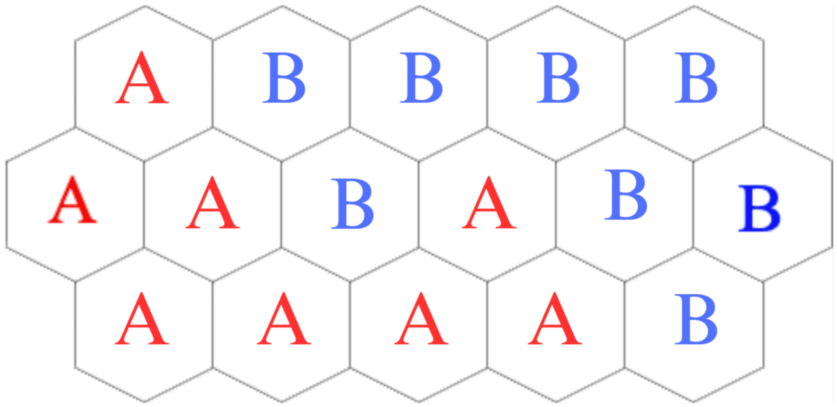
   $r = 4$: 1 solution

   $r = 5$: 2 solutions

   $r = 6$: 1 solution

   $r = 7$: 1 solution

   $r = 8$: 4 solutions: note this particularly tricky solution:



$r = 9$ is a mirror of $r = 7$, $r = 10$ is a mirror of $r = 6$, and so on.

Thus, the answer is $2(1 + 2 + 1 + 1) + 4 = \boxed{14}$.

3. Let O be the center of a unit circle and d be a diameter of this circle. Consider a point $X \neq O$ with $XO \perp d$. Let points $X_1, X_2, \ldots$ be the points on line XO such that $\overline{X_iO} = \frac{1}{2^i}$. Let p_i be the chord passing through X_i with $p_i \perp XO$ and let C_i and B_i be the endpoints of p_i. If A_i is the area of the triangle OB_iC_i, compute

$$\sum_{i=1}^{\infty} A_i^2.$$

Solution: Each triangle OB_iC_i is formed of two congruent right triangles with hypotenuse 1 and height $\frac{1}{2^i}$. The base of these triangles is therefore

$$\sqrt{1 - \left(\frac{1}{2^i}\right)^2} = \sqrt{1 - \frac{1}{2^{2i}}}$$

The area A_i is thus

$$\sqrt{1 - \frac{1}{2^{2i}}} \cdot \frac{1}{2^i} = \sqrt{\frac{1}{2^{2i}} - \frac{1}{2^{4i}}}$$

Therefore,

$$\begin{aligned}\sum_{i=1}^{\infty} A_i^2 &= \sum_{i=1}^{\infty} \frac{1}{2^{2i}} - \sum_{i=1}^{\infty} \frac{1}{2^{4i}} \\ &= \frac{1}{3} - \frac{1}{15} \\ &= \boxed{\frac{4}{15}}.\end{aligned}$$

4. Triangle ABC has $AB = 13, BC = 14, CA = 15$. A circle with the same radius as the incircle of $\triangle ABC$ is centered at vertex A of $\triangle ABC$. It is then slid along side AB to become centered at point B, then slid along slid BC to become centered at point C, and finally slid along CA to become centered at point A. Find the perimeter of the region swept out by the circle.

Solution: Let r be the inradius of $\triangle ABC$, and s the semiperimeter. Note that $s = 21$. By Heron's formula, the area of the triangle is 84, which by the formula $A = rs$ implies $r = 4$. Now, we claim that the region swept out by the circle has no interior region. Indeed, since $r = 4$ is the inradius, every slide along a side length passes through the same point, and together, they sweep through the entire triangle. To find the perimeter, it suffices to find the length of the exterior. Let ω_A, ω_B, and ω_C be the circles centered at A, B, and C, respectively, with the same radius as the inradius. Let l_1, l_2, l_3 be the common tangents between ω_A, ω_B, and ω_B, ω_C and ω_C, ω_A not intersecting the triangle. We see that lengths of these tangents are the side lengths of the triangle, giving a total length of $2s = 42$. Finally, the remainder of the exterior has three arcs add up to $180^\circ - \angle A + 180^\circ - \angle B + 180^\circ - \angle C = 360^\circ$, making one whole circle of radius four, contributing $2(4\pi) = 8\pi$ more to the perimeter. Putting it all together, the answer is $\boxed{42 + 8\pi}$.

5. There are 4 outlets A, B, C, D in a room, each with 2 sockets denoted with subscripts 1 and 2 (i.e. outlet A has sockets A_1 and A_2). A baby starts plugging wires into the sockets. They have 3 wires, and each wire has two ends that each plug into a socket. Each socket can take at most one plug.

   A short circuit happens when there is a loop from one outlet to itself. For example, $A_1 \leftrightarrow A_2$ is a short circuit and $A_1 \leftrightarrow B_1, B_2 \leftrightarrow C_1, A_2 \leftrightarrow C_2$ is a short circuit, but $A_1 \leftrightarrow B_1$ is not.

   Provided that the baby randomly connects every plug to a socket, what is the probability that the baby causes a short circuit?

   Solution: We do casework on the number of loops that are formed by the wires, and we count such that the wires are indistinguishable.

   For the case where 3 loops are formed, each wire must form its own loop. There are 4 ways to choose the outlets that have a loop from a single wire.

   For the case where 2 loops are formed, we either have two wires each forming its own loop, or one wire forming its own loop and two wires forming a loop together. In the case that two wires each form its own loop, there are $\binom{4}{2} = 6$ ways to choose the sockets for those wires. Then, the third wire must not form a loop, so it can choose a socket from each outlet remaining, and there are 4 ways to do so (e.g. $\{C_1, C_2\} \times \{D_1, D_2\}$), so we get $6 \times 4 = 24$. In the case that one wire forms a loop and the other two wires form a loop together, there are 4 ways to choose the outlet that has a loop with itself, 3 ways to choose the outlet that does not connect with any wire, and 2 ways to choose how the sockets from the two outlets in a loop are paired up, so we have $4 \times 3 \times 2 = 24$.

   For the case where 1 loop is formed, the loop involves either 1, 2, or 3 wires. If 1 wire is involved, there are 4 ways to choose the outlet for it, and in order for the other two wires to not form a loop there must be an outlet connected to both wires, with each wire going to a different one of the remaining two outlets (e.g. $B_1 \leftrightarrow C_1, B_2 \leftrightarrow D_1$) There are 3 ways to choose which outlet has both sockets connected, 4 ways to choose which socket the first socket connects to, and 2 ways to choose which the second socket connects to, so we have $4 \times 3 \times 4 \times 2 = 96$. If 2 wires are involved in the loop, there are $\binom{4}{2} = 6$ ways to choose which two outlets are in the loop and 2 ways to choose how the sockets within the loop are connected. The third wire has 4 ways to not form a loop, so we have $6 \times 2 \times 4 = 48$. If 3 wires are involved in the loop, there are 4 ways to choose which outlet is not used. Suppose $A_1, A_2, B_1, B_2, C_1, C_2$ are used in the loop. Then, there are 4 options for the socket A_1 connects to and then 2 options for which socket A_2 connects to (the rest of the loop is then determined), so we have $4 \times 4 \times 2 = 32$.

Adding all of the cases gives us 228, and the total number of ways to choose how the sockets are connected is $\frac{\binom{8}{2}\binom{6}{2}\binom{4}{2}}{3!} = 420$, so our answer is $\frac{228}{420} = \boxed{\frac{19}{35}}$.

6. Point P lies inside rectangle $ABCD$ so that $PA = 52$, $PB = 60$, and $PD = 25$. Let $\overrightarrow{AP}$ intersect CD at A', $\overrightarrow{BP}$ intersect CD at B', and $\overrightarrow{CP}$ intersect AD at C'. If $AB = 56$, what is the ratio of the area of $\triangle PA'B'$ to the area of $\triangle PC'D$?

   Solution: Let the feet of the altitudes from P to AB and CD be X and X', respectively. By the Pythagorean theorem in $\triangle PAB$, we have $60^2 - BX^2 = 52^2 - AX^2$. We also know that $AX + BX = AB = 56$. Solving the system of equations gives $AX = 20$ and $BX = 36$. Also, $PX = \sqrt{52^2 - 20^2} = 48$ and $PX' = \sqrt{PD^2 - DX'^2} = \sqrt{25^2 - 20^2} = 15$. Note that $\triangle PAB$ and $\triangle PA'B'$ are similar, so $A'B' = \frac{15}{48} \cdot 56 = \frac{35}{2}$. The area of $\triangle PA'B'$ is $\frac{1}{2}(PX')(A'B') = \frac{1}{2} \cdot 15 \cdot \frac{35}{2}$. Now, let the feet of the altitudes from P to BC and AD be Y and Y', respectively. We see that $\triangle C'CD$ is similar to $\triangle CPX'$, so $C'D = \frac{56}{36}PX' = \frac{56}{36} \cdot 15 = \frac{70}{3}$. The area of $\triangle PC'D$ is $\frac{1}{2}(PY')(C'D) = \frac{1}{2} \cdot 20 \cdot \frac{70}{3}$. So, $\frac{[PA'B']}{[PC'D]} = \frac{15 \cdot \frac{35}{2}}{20 \cdot \frac{70}{3}} = \boxed{\frac{9}{16}}$.

7. There are two birds currently sitting on a tree. At the end of every passing hour, there is a $1 - \frac{1}{n}$ probability that one of the birds on the tree at that time leaves the tree (where n is the number of birds sitting on the tree that past hour). Similarly, at the end of every passing hour there is an independent probability of $\frac{1}{n}$ that one new bird will come sit on the tree. What is the expected number of hours that will pass until there are four birds on the tree for the first time?

   Solution: For $n = 1$, the probability that a bird will arrive is 1, and the probability that a bird will leave is 0. For $n = 1, 2, 3$, we have that the probability that the number of birds increases by 1 is $(1/n)^2$, since not only does a new bird have to arrive but also no birds can leave for this to happen. Similarly, the probability that the number of birds decreases by 1 at the end of the hour is $(1 - 1/n)^2$. The probability that the number of birds stays the same after the hour passes is $2(1/n)(1 - 1/n)$.

   Let $S_{a,b}$ denote the expected number of hours that will pass so that a birds on the tree will become b birds on the tree, where $a \leq b$. Then we have the following equations:

$$S_{1,4} = 1 + S_{2,4}$$

$$S_{2,4} = \frac{1}{4}(1 + S_{1,4}) + \frac{1}{4}(1 + S_{3,4}) + \frac{1}{2}(1 + S_{2,4})$$

$$S_{3,4} = \frac{1}{9} + \frac{4}{9}(1 + S_{3,4}) + \frac{4}{9}(1 + S_{2,4})$$

   We can solve these (substitute $S_{1,4}$ away to get two equations with two variables) to get $S_{2,4} = \boxed{34}$.

8. Find the number of ordered pairs of positive integers (a, b) such that $a^2 + b^2$ is a divisor of 2024^2.

Solution: First off, $2024 = 2^3 \cdot 11 \cdot 23$. To demand that $a^2 + b^2 \mid 2^6 \cdot 11^2 \cdot 23^2$, the prime factorization of $a^2 + b^2$ must consist of precisely $2, 11, 23$. For each of 11 and 23 note that we must have either a, b both divisible by it, or neither. This allows us to neatly partition the set of of (a, b) into four: $(a, b), (11a, 11b), (23a, 23b), (253a, 253b)$, which are mutually disjoint.

So now we consider the number of (a, b) such that $a^2 + b^2 \mid 64$. Consider the order of 2 in $a^2 + b^2$, when a, b do not share 2 as a divisor. If a and ba are both odd, then the sum of their squares is equal to $2 (\text{mod } 4)$, so the degree is simply 1. When one of a and b is odd and the other is even, the sum of squares is odd and must contain prime divisors other than 2 (unless a or b is 0, and the other is 1). After dividing a and b by the largest power of 2 that divides both of them, we see that we end up with either two odd numbers or an odd and even number. We have already discarded the latter case, and in the former case the sum is $2 (\text{mod } 4)$, so the numbers must both be 1 in order to have a power of 2. We have shown that if $a^2 + b^2 \mid 64$, then $a = b = 2^n$ for some n.

Thus, $a^2 + b^2 = 2^{2n+1}$. Since this must divide $64 = 2^6$, we have $n = 0, 1$, or 2. So, the number of pairs that work overall is $3 \cdot 4 = \boxed{12}$.

9. Dean's life is going in circles. Help him escape! Dean is standing in the center of a unit circle. At each moment he can only go east or north-east (at a 45° angle). He walks until he reaches the circumference of the circle. Compute the square of the length of the longest path he could take.

Solution: Each path has an equivalent path of the same length which first goes x East (without changing direction) and then goes y North-East (without changing direction). We can decompose the North-East movement into its North and East components giving us the equation $\left(x + \frac{1}{\sqrt{2}}y\right)^2 + \left(\frac{1}{\sqrt{2}}y\right)^2 = (x)^2 + \frac{2}{\sqrt{2}}xy + (y)^2 = 1$. We are trying to maximize $(x + y)^2 = (x)^2 + 2xy + (y)^2$. Combining these two equations gives us that we are trying to maximize $\left(x^2 + \frac{2}{\sqrt{2}}xy + y^2\right) + \left(2 - \frac{2}{\sqrt{2}}\right)xy = 1 + \left(2 - \frac{2}{\sqrt{2}}\right)xy$. Therefore, we need to maximize xy. Notice that $(x - y)^2 = x^2 - 2xy + y^2 = \left(x^2 + \frac{2}{\sqrt{2}}xy + y^2\right) - \left(2 + \frac{2}{\sqrt{2}}\right)xy = 1 - \left(2 + \frac{2}{\sqrt{2}}\right)xy$. We know that $(x - y)^2 \geq 0$, so $1 - \left(2 + \frac{2}{\sqrt{2}}\right)xy \geq 0$. Therefore, the maximum of xy (which can be achieved through any positive x and y with this product) is $xy = \frac{1}{2 + \frac{2}{\sqrt{2}}}$. Therefore, the maximum of the square of the length of the longest path is $1 + \left(2 - \frac{2}{\sqrt{2}}\right)\left(\frac{1}{2 + \frac{2}{\sqrt{2}}}\right) = \boxed{4 - 2\sqrt{2}}$.

10. A group of scientists are researching the population dynamics of a certain bacterial colony. They find that if P_n represents the bacterial population at n minutes from the start, then this population can be modeled in a surprisingly discrete way given by $P_n = \frac{P_{n-1}^3}{P_{n-3}^4}$. Given that in one experiment, $P_0 = 1, P_1 = 1$, and $P_2 = 7^3$, find $3 \log_7(P_{100})$.

Solution: Taking logs of the equation $P_n = \frac{P_{n-1}^3}{P_{n-3}^4}$ gives: $\log P_n = 3 \log P_{n-1} - 4 \log P_{n-3}$. Note that the base of the logarithm is not specified yet. Now, we can consider the sequence $l_n = \log P_n$. We get the nice recurrence relation: $l_n = 3l_{n-1} - 4l_{n-3}$. This can be solved by considering that l_n takes the form of r^n. We then get: $r^n = 3r^{n-1} - 4r^{n-3} \implies r^3 - 3r^2 + 4 = 0 \implies r_1, r_2 = 2, r_3 = -1$. Due to repeated roots we get $l_n = a2^n + bn2^n + c(-1)^n$ for some constants a, b, c. Initial conditions give us $l_0 = 0 \implies a + c = 0$, $l_1 = \log P_1 = 2a + 2b - c$, and $l_2 = \log P_2 = 4a + 8b +$

c. Now we can invoke a base of the logarithm and choose it to be base 7 and solve the system of equations to get, to get $(a, b, c) = (-\frac{1}{3}, \frac{1}{2}, \frac{1}{3}) \implies l_n = -2^n/3 + n2^{n-1} + (-1)^n/3 \implies P_n = 7^{-2^n/3+n2^{n-1}+(-1)^n/3}$. Plugging in $n = 100$ gives the desired result of $\boxed{1 + 149 \cdot 2^{100}}$.

11. Floria is picking flowers on a vast prairie with six different colors of flowers, where each color is equally likely to be picked at random, independent of previously picked flowers. Out of the flowers she picks, Floria wants to make a bouquet of 7 flowers, such that all 6 colors are used and there is a pair of identically colored flowers. If she picks flowers, one by one, at random, what is the expected number of flowers Floria needs to pick in order to make the bouquet?

Solution: Let X_i for $i = 1, ..., 6$ denote the number of extra flowers Floria needs to pick in order to get i distinct flower colors, respectively, after already obtaining $i - 1$ distinct colors of flowers. Let Y denote the number of extra flowers Floria needs to pick in order to get a pair of the same color after getting 6 distinct colors. Note that we want

$$\mathbb{E}[X_1 + \cdots + X_6 + Y]$$
$$= \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_6] + \mathbb{E}[Y].$$

Now, for each $i = 2, ..., 6$, note that when Floria picks a flower, we have a $\frac{7-i}{6}$ chance of choosing a new flower color, and a $\frac{i-1}{6}$ chance of not choosing a new flower color and being back in the same state. Thus we have

$$\mathbb{E}[X_i] = 1 + \frac{7-i}{6} \cdot 0 + \frac{i-1}{6} \cdot \mathbb{E}[X_i] = 1 + \frac{i-1}{6} \cdot \mathbb{E}[X_i],$$

which upon solving gives

$$\mathbb{E}[X_i] = \frac{6}{7-i},$$

and now noting that $\mathbb{E}[X_1] = 1$, it follows that

$$\mathbb{E}[X_1] + \cdots + \mathbb{E}[X_6]$$
$$= \frac{6}{1} + \frac{6}{2} + \frac{6}{3} + \frac{6}{4} + \frac{6}{5} + \frac{6}{6} = \frac{147}{10}.$$

Note that given six distinct colors, the only way we do not have a pair of the same color is if the first six colors picked are all distinct. This occurs with probability $\frac{6!}{6^6} = \frac{5}{324}$. Thus $Y = 1$ with $\frac{6!}{6^6} = \frac{5}{324}$ probability, and otherwise $Y = 0$. It follows

$$\mathbb{E}[Y] = 1 \cdot \frac{5}{324} + 0 \cdot \left(1 - \frac{5}{324}\right) = \frac{5}{324},$$

and the answer is

$$\frac{147}{10} + \frac{5}{324} = \boxed{\frac{23839}{1620}}.$$

12. Suppose ABC is a triangle with $AB = 13$, $BC = 15$, and $AC = 14$. Let H be the orthocenter of $\triangle ABC$, and let M be the midpoint of BC. If P is the intersection of the circumcircle of $\triangle ABC$ with line MH that lies on minor arc AB, compute the length of HP.

    Solution: Denote the circumcenter of $\triangle ABC$ as O. We show that that AO and MH intersect on (ABC). Let H' be the reflection of H in M. We see that BC and HH' bisect each other, so $HBH'C$ is a parallelogram. Then, $\angle BH'C = \angle BHC = 180^\circ - (90^\circ - \angle C) - (90^\circ - \angle B) = 180^\circ - \angle A$, so H' lies on the circumcircle of $\triangle ABC$. Let H'' be the reflection of H over line BC. Note that $\angle BH''H = \angle BHH'' = 90^\circ - (90^\circ - \angle C) = \angle C$, so H'' also lies on (ABC). Note that H' and H'' have the same distance from BC, so $H'H'' \parallel BC$. We now see that $\triangle AH''H'$ is a right triangle, so AH' is a diameter. Thus, it follows that AHP is a right triangle with $\angle HPA = 90^\circ$.

    Let D be the foot of A on BC. Then, we have $\angle HDM$ is also a right angle. Thus, $APDM$ is cyclic, and $HP = \frac{AH \cdot HD}{HM}$ by Power of a Point. Let E be the foot of B on AC. We have $AE = 5$ and $\sin \angle EAH = \frac{3}{5}$ (from angle chasing), so $AH = \frac{25}{4}$ and $EH = \frac{15}{4}$. Then, $BH = 12 - EH = \frac{33}{4}$. Since $\sin \angle HBD = \frac{3}{5}$, we have $HD = \frac{99}{20}$ and $BD = \frac{33}{5}$. Then, $DM = BM - BD = \frac{9}{10}$. We have $HM = \sqrt{(HD)^2 + (DM)^2} = \frac{9\sqrt{5}}{4}$.

    Our final answer is $HP = \frac{\frac{25}{4} \cdot \frac{99}{20}}{\frac{9\sqrt{5}}{4}} = \boxed{\frac{11\sqrt{5}}{4}}$.

13. Two parabolas $y = (x-1)^2 + a$ and $x = (y-1)^2 + b$ intersect at a single point, where a and b are non-negative real numbers. Let c and d denote the minimum and maximum possible values of $a + b$, respectively. Compute $\lfloor c \rfloor + d$.

    Solution: The first and most crucial step is to realize that we can replace either one of the equations with $(x-1)^2 - x + (y-1)^2 - y + a + b = 0$. Completing squares, we get $\left(y - \frac{3}{2}\right)^2 + \left(x - \frac{3}{2}\right)^2 = \frac{5}{2} - (a+b)$, which defines a circle. The nice thing about this circle is that its center is fixed, and its radius depends only on $a + b$.

    We claim that the maximum possible value of $a + b$ is $\frac{5}{2}$, and the minimum is attained when either a or b is 0. For the maximum, we can see that in order for the parabolas to intersect, there must be at least one point on the circle we have found, so we must have $a + b \le \frac{5}{2}$. When $a + b = \frac{5}{2}$, we must have $x = y = \frac{3}{2}$, so the maximum is attained by setting $a = b = \frac{5}{4}$. There is only one point of intersection of the parabolas with these values.

    To find the minimum possible value of $a + b$, it is good to draw some pictures. Since there is symmetry, we may assume WLOG that $\left(\frac{3}{2}, \frac{3}{2}\right)$ lies above the parabola $y = (x-1)^2 + a$, i.e. $a \le b$.

    Fix such a value of a, and consider what happens as b increases from 0 to the value when the two parabolas are tangent. The circle shrinks, the vertical parabola stays constant, while the horizontal parabola shifts right. At the point of tangency, the circle retracts to the point that is interiorly tangent to the vertical parabola, and exteriorly tangent to the horizontal one. Therefore, the value of $a + b$ for this particular value of a is entirely determined by the radius of this circle that would make it tangent to a fixed parabola. This radius value increases as a is decreases—if a is smaller, then the parabola is further away from $\left(\frac{3}{2}, \frac{3}{2}\right)$.

    The minimum of $a + b$ occurs when a or b vanishes; let us say $a = 0$. Then, we seek the number b such that $\left(y - \frac{3}{2}\right)^2 + \left(x - \frac{3}{2}\right)^2 = \frac{5}{2} - b$ is tangent to $y = (x-1)^2$. While finding the precise value of b is very painful and requires calculus, finding the floor is easier. Simply observe that if $b = 2$ there is at least one intersection: a circle centered at $\left(\frac{3}{2}, \frac{3}{2}\right)$ of radius $\sqrt{1/2}$ clearly intersects $y = (x-1)^2$ at $(2, 1)$, so $b \geq 2$. By the previous discussion of the maximum, the floor cannot be $3 > \frac{5}{2}$, so the floor is 2.

Therefore, answer is $\frac{5}{2} + 2 = \boxed{\frac{9}{2}}$.

14. Let $f(n)$ be the number of positive divisors of n that are of the form $4k+1$, for some integer k. Find the number of positive divisors of the sum of $f(k)$ across all positive divisors of $2^8 \cdot 29^{59} \cdot 59^{79} \cdot 79^{29}$.

Solution: Let $N = 2^8 \cdot 29^{59} \cdot 59^{79} \cdot 79^{29}$. Additionally, let $g(n)$ be the number of divisors of n that are $3 \pmod 4$.

Suppose that a, b are relatively prime. Then,

$$\begin{aligned}(f(a) - g(a))(f(b) - g(b)) &= (f(a)f(b) + g(a)g(b)) - (f(a)g(b) + g(a)f(b)) \\ &= f(ab) - g(ab).\end{aligned}$$

Thus, we have that $f(n) - g(n)$ is multiplicative for relatively prime integers. Let this function be $h(n)$.

We can compute the values that h obtains on powers of primes:

- When $n = 2^k$, $h(n) = 1$.
- When $n = p^k$ with $p \equiv 1 \pmod 4$, then $h(n) = k + 1$.
- When $n = p^k$ with $p \equiv 3 \pmod 4$, then $h(n) = 1$ if k is even, and $h(n) = 0$ when k is odd.

Additionally, we know that $\sigma(p^k) = k + 1$ for an odd prime p, where $\sigma(n)$ is the number of odd divisors of n. Note that $\sigma(n) = f(n) + g(n)$ and is also multiplicative with respect to relatively prime integers.

It therefore follows that

$$\begin{aligned}2\sum_{d \mid N} f(d) &= \sum_{d \mid N} h(d) + \sum_{d \mid N} \sigma(d) \\ &= \left(\sum_{d \mid 2^8} h(d)\right)\left(\sum_{d \mid 29^{59}} h(d)\right)\left(\sum_{d \mid 59^{79}} h(d)\right)\left(\sum_{d \mid 79^{29}} h(d)\right) \\ &+ \left(\sum_{d \mid 2^8} 1\right)\left(\sum_{d \mid 29^{59}} \sigma(d)\right)\left(\sum_{d \mid 59^{79}} \sigma(d)\right)\left(\sum_{d \mid 79^{29}} \sigma(d)\right) \\ &= 9 \cdot \frac{60 \cdot 61}{2} \cdot \frac{80}{2} \cdot \frac{30}{2} + 9 \cdot \frac{60 \cdot 61}{2} \cdot \frac{80 \cdot 81}{2} \cdot \frac{30 \cdot 31}{2} \\ &= 9 \cdot 30 \cdot 61 \cdot 40 \cdot 15 \cdot (81 \cdot 31 + 1) \\ &= 2^8 \cdot 3^4 \cdot 5^3 \cdot 61 \cdot 157.\end{aligned}$$

The sum of the divisors factored is $2^7 \cdot 3^4 \cdot 5^3 \cdot 61 \cdot 157$, which has $8 \cdot 5 \cdot 4 \cdot 2 \cdot 2 = \boxed{640}$ divisors.

15. Triangle ABC has side lengths $AB = 24$ and $BC = 23$, and is inscribed in the circle ω. The radius of ω is 15 and point P lies on minor arc BC of ω. Let M be the midpoint of AB. Line PM intersects ω at $F \neq P$. Let T be the intersection of the tangents to ω passing through A and B. Let TF intersect AB at K, and ω at $L \neq F$. Finally, let FC intersect BP at Q, and let TQ intersect BC at N. If $TL = 25$, compute the area of $\triangle KMN$.

Solution: We note that $[APM] = [BPM]$ and $[AFM] = [BFM]$, so $[APF] = [BPF]$. Also, $\sin\angle PAF = \sin\angle PBF$, and we conclude by the Law of Sines that $(AF)(AP) = (BF)(BP)$. Let L' be the reflection of P over line TM. Then, $ABPL'$ is an isosceles trapezoid, so it is a cyclic quadrilateral. Also, $AP = BL'$ and $AL' = BP$, so we have $(AF)(BL') = (BF)(AL')$. This means that $AFBL'$ is a harmonic quadrilateral, which has the property that the tangents to its circumcircle at the endpoints of a diagonal either intersect on the other diagonal or are parallel. This means that T lies on line $L'F$. We can then conclude that in fact $L' = L$, so we have shown that $PL \parallel AB$.

Let N' be the intersection of BC and the line passing through P parallel to AB. By Pascal's Theorem on $PBBCFL$ (note that BB indicates the tangent at B), we see that Q, N', and T are collinear. Since N' is the intersection of QT and BC, we conclude that $N' = N$. This means that N lies on PL.

We now proceed to find the area of $\triangle KMN$. Let O be the center of ω. We have $OA = 15$ and $AM = 12$, so $OM = 9$. We note that $\triangle OAM \sim \triangle OTA$, from which we find that $TA = 20$. Since $TL = 25$, by Power of a Point we have $(TF)(25) = 400$, which gives us $TF = 16$. Also from the similar right triangles, $TM = 16$ and $TO = 25$. Let the foot of the perpendicular from F to TO be J. We see that $(FJ)^2 + (TJ)^2 = 16^2$ and $(FJ)^2 + (25 - TJ)^2 = 15^2$, which gives us $TJ = \frac{328}{25}$. Let PL intersect line TM at I. Note that $\triangle FJT \sim \triangle LIT$ with ratio $\frac{16}{25}$, so $TI = \frac{328}{25} \cdot \frac{25}{16} = \frac{41}{2}$. We find that $LI = \frac{1}{2}\sqrt{50^2 - 41^2} = \frac{3\sqrt{91}}{2}$, so $[TPL] = \frac{41 \cdot 3\sqrt{91}}{4}$. We have $[LPF] = [TPL] \cdot \frac{LF}{LT} = \frac{41 \cdot 3\sqrt{91}}{4} \cdot \frac{9}{25}$. We see that $\triangle IMP \sim \triangle JMF$ with ratio $\frac{IM}{MJ}$. We have $MJ = TM - TJ = \frac{72}{25}$ and $IM = IT - MT = \frac{9}{2}$, so the ratio is $\frac{25}{16}$. Compared to $\triangle LPF$, for $\triangle KMN$ we have $KM = \frac{16}{16+25}(LP)$, and the height is $1 - \frac{16}{16+25} = \frac{25}{41}$ times that of $\triangle LPF$. Thus, we have

$$[KMN] = \frac{41 \cdot 3\sqrt{91}}{4} \cdot \frac{9}{25} \cdot \frac{16}{41} \cdot \frac{25}{41} = \boxed{\frac{108\sqrt{91}}{41}}.$$