1 What is Social Choice?

Everyone has things that they want. You might like apples, while your SMT teammates might like bananas. How should you decide who gets what items?

One approach is through prices: At a grocery store, those who are willing to pay the listed price for apples will purchase apples, and those who are willing to pay the listed price for bananas will purchase bananas. Yet in many cases, prices don't work too well.

For instance, if we're trying to decide who should next receive a kidney donation or who the next President should be, using a market-based mechanism would attract much criticism. In these situations, the science of how to aggregate people's diverse wants becomes important. This Power Round will introduce some of the mathematics behind how societies make decisions.

2 Preferences

Preferences can be represented by a *binary relationship*, which we will eventually develop into a *partial order*. A natural order that you are probably familiar with is "$\geq$" over the real numbers. For example, consider a set G of all items in a grocery store. You might like apples more than oranges or be indifferent between the two fruits, which can be represented as "apple $\succeq$ orange". If this is the case, we say that apples are weakly preferred to oranges. By convention, everything is weakly preferred to itself, so "apple $\succeq$ apple". If oranges are also weakly preferred to apples, then "orange $\succeq$ apple" as well. In this case, you would be *indifferent* between apples and oranges and like them the same. In general, if $x \succeq y$ and $y \succeq x$, we can write that $x \sim y$.

Problem 2.1 (1pt). *Is indifference a symmetric relationship? That is, if $x \sim y$, then is it necessarily the case that $y \sim x$?*

> Yes. Suppose $x \sim y$. Then, $x \succeq y$ and $y \succeq x$. By simply moving statements around, $y \succeq x$ and $x \succeq y$. Thus, $y \sim x$.

What happens if apples are weakly preferred to oranges, but oranges are *not* weakly preferred to apples, so "apple $\succeq$ orange" but "orange $\not\succeq$ apple"? In this case, there is no longer indifference between apples and oranges, and we denote "apple $\succ$ orange". In general, if $x \succeq y$ and $y \not\succeq x$, we can write that $x \succ y$ and say that x is *strictly* preferred to y.

Without any additional assumptions, binary relationships are not too interesting. For example, someone can walk into a grocery store and not have any thoughts about what they like at all. To rule such trivial cases out, a common assumption is that binary relationships are *complete.*

Definition 2.1. *A binary relation $\succeq$ over a set G is complete if for all elements x and y in G, either $x \succeq y$ or $y \succeq x$ (or both).*

Similarly, another common assumption is transitivity: If apples are better than oranges, and oranges are better than bananas, then apples should be better than bananas as well.

Definition 2.2. *A binary relation $\succeq$ over a set G is transitive if $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.*

Problem 2.2 (6 pts). *Suppose $\succeq$ over a finite set G is complete and transitive. Show that G has a maximal element: There exists some x in G such that for any y in G, we have that $x \succeq y$.*

> Sol 1. We show this by induction on $|G|$. If $|G| = 1$, then let $G = \{x\}$. Clearly, $x \succeq x$ so x is maximal in G. Next, suppose the result holds for sets of size n, and consider $|G| = n + 1$. Pick any element of G, say y. We have that $|G \setminus \{y\}| = n$, so the inductive hypothesis gives that there is a maximal element g of $G \setminus \{y\}$. By completeness, either $g \succeq y$ or $y \succeq g$. If $g \succeq y$, then g is maximal. If $y \succeq g$, then $g \succeq z$ for all z in $G \setminus \{y\}$ and transitivity gives that $y \succeq z$ for all z in G, and y is maximal.
> Sol 2. Proof by contradiction. ASFOC there is no maximal element. Start with arbitrary element x_1; then there exists a chain of elements such that $x_n \succeq x_{n+1}$ does not hold. By completeness, it then must be that $x_{n+1} \succeq x_n$. The set being finite, there must be repetition, i.e. for some $m > n$, $x_n = x_m$. But then via transitivity, $x_n \succeq x_{n+1}$, contradiction.

Problem 2.3 (2 pts). *Give an example of a complete and transitive binary relation over an infinite set such that no maximal element exists.*

One example would be $\geq$ over the integers. For any integer n, it is not true that $n \geq n+1$. Any example that works receives full credit.

Problem 2.4 (4 pts). *Suppose $\succeq$ is a transitive binary relation. Is $\succ$ a transitive binary relation? Prove or provide a counterexample.*

Yes, $\succ$ is transitive. We will show that $a \succ b$ and $b \succ c$ implies $a \succ c$.
Suppose $a \succ b$ and $b \succ c$. Then, $a \succeq b$ and $b \succeq c$ so $a \succeq c$. To show that $a \succ c$, it suffice to show that c is not weakly preferred to a.
Towards a contradiction, suppose $c \succeq a$. As such, $c \succeq a$ and $a \succeq b$ implies $c \succeq b$ by transitivity. However, this contradicts $b \succ c$ since $b \succ c$ implies $c \not\succeq b$.

2.1 Utility Function Representations

In general, it is very difficult to work with partial orders directly when defining what someone's preferences are. For instance, if a set has n elements in it, requiring completeness requires at least $\binom{n}{2} + n = \frac{n(n+1)}{2}$ comparisons, a number that grows quickly.

Problem 2.5 (2 pts). *What is the maximum number of weak preferences a partial order/binary relation (that may not necessarily be complete or transitive) may have over n elements?*

Suppose everything is weakly preferred to anything else. Then, there are n^2 weak preferences (since every element is also weakly preferred to itself).

Instead, a useful representation of preferences is through utility functions. Suppose your preferences over $\{x, y, z\}$ was $x \succeq y, y \succeq z, x \succeq z$. One way to represent this is by defining

$$u(x) = 3, u(y) = 2, u(z) = 1$$

and to say that $a \succeq b$ if and only if $u(a) \geq u(b)$.

Definition 2.3 (Utility Function). *Given preferences $\succeq$ over the set G, a utility function u from G to real numbers represents $\succeq$ if for any a, b in G, we have that $a \succeq b$ if and only if $u(a) \geq u(b)$.*

A useful property is that if preferences can be represented by a utility function, natural properties are automatically satisfied. For instance:

Problem 2.6 (4 pts). *Suppose $\succeq$ on G can be represented by the utility function u. Show that $\succeq$ is complete and transitive.*

Complete. Given a, b we have that $u(a)$ and $u(b)$ are real numbers. Then, for any two numbers, either $u(a) \geq u(b)$, in which case $a \succeq b$, or $u(b) \geq u(a)$, in which case $b \succeq a$.
Transitive. Suppose $a \succeq b$ and $b \succeq c$. Then $u(a) \geq u(b)$ and $u(b) \geq u(c)$. Thus, $u(a) \geq u(b) \geq u(c)$ so $a \succeq c$ as desired.

In general, there can be many utility functions that represent the same preferences.

Definition 2.4. *A function f is monotonic if $x > y$ implies $f(x) > f(y)$.*

Problem 2.7 (4 pts). *Suppose the utility function g represents preferences $\succeq$. Show that if f monotonic, then $f \circ g$ also represents preferences $\succeq$, where $f \circ g$ is the function f composed with g: $(f \circ g)(x) = f(g(x))$.*

We have that $a \succeq b$ if and only if $g(a) \geq g(b)$ as g represents $\succeq$. Then, $f(g(a)) \geq f(g(b))$ if and only if $g(a) \geq g(b)$ as f is monotonic. Thus, $a \succeq b$ if and only if $f(g(a)) \geq f(g(b))$.

3 Voting Systems

An important yet difficult problem plaguing society for years is that of preference aggregation: Given a set of individual preferences over some set of different outcomes, how can we aggregate that into some democratic societal ranking over outcomes?

Suppose you and your teammates want to celebrate surviving the Stanford Math Tournament power round. There are many choices that need to be made: Where should you go? When should the celebration take place?

In general, societies have always struggled with preference aggregation. Even very natural aggregation schemes fail to preserve the desired properties of (individual) preference orderings. Suppose there are three individuals, Alice, Bob, Carl, and three possible outcomes, x, y, z. To simplify notation, we write preferences in the notation of *linear orders*: We list outcomes in a way such that the first outcome is the most-preferred, the second outcome is the second most-preferred, and so on. Linear orders induce complete and transitive preferences. For instance, let Alice, Bob, and Carl's preferences be:

- Alice: $x \succ y \succ z$;
- Bob: $z \succ x \succ y$;
- Carl: $y \succ z \succ x$.

Suppose preferences are aggregated by the following rule: Society has $a \succ b$ if and only if more individuals have $a \succ b$ than $b \succ a$. So for instance, society has $x \succ y$ since Alice and Bob have $x \succ y$, but only Carl has $y \succ x$.

Problem 3.1 (2 pts). *Show that with these individual preferences, the given aggregation rule fails transitivity.*

As noted, $x \succ y$. Then, Alice and Carl both have $y \succ z$, so society has $y \succ z$ as well. If society's preferences were transitive, this then implies that $x \succ z$. However, only Alice has $x \succ z$ while Bob and Carl have $z \succ x$, so $z \not\succ x$, a violation of transitivity.

Similarly, it could be the case that a reasonable aggregation scheme fails to produce an outcome at all.

Definition 3.1. *An outcome x is a Condorcet winner if more individuals have $x \succeq y$ than $y \succeq x$ for all y.*

A Condorcet winner does not always exist. Consider the preferences Alice, Bob, and Carl's preferences from before. None of the outcomes are Condorcet winners:

- Alice and Bob have $x \succ y$ so y cannot be a Condorcet winner;
- Alice and Carl have $y \succ z$ so z cannot be a Condorcet winner;
- Bob and Carl have $z \succ x$ so x cannot be a Condorcet winner.

Problem 3.2 (2 pts). *Construct another example of preferences for which no Condorcet winner exists, and show why no Condorcet winner exists.*

Suppose we add an "irrelavent" outcome I to the previous example, so preferences are

- Alice: $x \succ y \succ z \succ I$;
- Bob: $z \succ x \succ y \succ I$;
- Carl: $y \succ z \succ x \succ I$.

The same reasoning as before gives that x, y, z cannot be a Condorcet winner. Clearly I is also not a Condorcet winner (it loses against any other outcome). Any example that works receives full credit.

Problem 3.3 (4 points). *Show that if the set of possible outcomes is finite, then either a Condorcet winner exists or there is a sequence of outcomes $a_1, ..., a_n$ such that weakly more people prefer a_i to a_{i+1} for every $i = 1, 2, ..., n-1$ and more people prefer a_n to a_1 (in other words, there is a cycle).*

Suppose there is no Condorcet winner. Then, for every outcome x, there exists an outcome y such that weakly more people prefer y to x. If this is the case, say that outcome x "loses" to outcome y.
Start at any outcome a_0. It must lose to some other outcome, so pick any arbitrary outcome a_0 loses to and call it a_1. Then, pick a_2 to be any outcome a_1 loses to. Inductively proceed in this manner. As the set of outcomes is finite, a cycle must eventually be formed.

Yet despite (or perhaps, because of) these difficulties, there can still be a rich theory of how societies aggregate preferences.

For instance, a classic result is the median voter theorem. Suppose an odd number of SMT problem writers are trying to decide on the number of boba teas they should buy at the next problem writing meeting. Each problem writer has an ideal number of bobas, and strictly prefers b bobas to b' bobas if and only if b is closer to their ideal number of bobas than b'. If there are ties, the voter will prefer the higher number of bobas. For instance, a problem writer with an ideal number of 10 bobas will have preferences over $\{0, 1, ..., 20\}$ of:

$$10 \succ 11 \succ 9 \succ 12 \succ 8 \succ 13 \succ 7 \succ 14 \succ 6 \succ 15 \succ 5 \succ 16 \succ 4 \succ 17 \succ 3 \succ 18 \succ 2 \succ 19 \succ 1 \succ 20 \succ 0.$$

Problem 3.4 (2 pts). *Suppose there are 21 problem writers, one for each ideal number of bobas from* 0 *to* 20*. Erick proposes purchasing* 7 *bobas and Vicktor proposes purchasing* 18 *bobas. How many problem writers vote for each proposal, and which proposal wins the majority of votes?*

Everyone with an ideal number of bobas less than or equal to 12 votes for Erick and everyone with an ideal number of bobas more than or equal to 13 votes for Vicktor. As such, 13 problem writers vote for Erick and 8 problem writers vote for Vicktor. Thus, Erick's proposal wins the majority of votes.

Problem 3.5 (4 pts). *Once again, suppose there are 21 problem writers with one at each ideal number of bobas from* 0 *to* 20*. Show that a proposal of purchasing* 10 *bobas will always get more votes than any other proposal.*

Suppose the other proposal has 9 or less bobas. Then, everyone whose preferred number of bobas is 10 or more votes for the proposal of 10 bobas, which is 11 votes and hence majority. Similarly, if the other proposal has 11 or more bobas, then everyone whose preferred number of bobas is 10 or less votes for the proposal of 10 bobas. Once again, this constitutes 11 votes, a majority.

This constitutes the reasoning behind the median voter theorem. Suppose there is any odd number of problem writers, and each of them has an arbitrary ideal number of bobas. (In particular, each problem writer's ideal number of bobas no longer has to be between 0 and 20, but they still prefer numbers closer to their ideal over numbers further from their ideal.) Define the median problem writer to be the problem writer with the median ideal number of bobas out of all problem writers. The median voter theorem says that, given two proposed number of bobas, the number that the median problem writer likes more will win a majority of votes.

Problem 3.6 (6 pts). *Prove this version of the median voter theorem.*

Fix any two outcomes. Let a be the outcome the median problem writer chooses and let b be the other outcome. Let m be the median problem writer's ideal amount of boba. It cannot be that $m = b$, as then the median problem writer would have chose b instead of a. There are two cases: either $m > b$ or $m < b$.
First, suppose $m > b$. Then, it must be that $a > b$ as m chooses a over b. Next, consider any problem writer with an ideal quantity q more than or equal to m. We have that

$$|q - a| \leq |a - m| + |m - q| \leq |b - m| + q - m = m - b + q - m = q - b = |q - b|$$

so q is closer to a than b and the problem writer chooses a. Thus, all problem writers with ideal quantities less than or equal to m choose outcome a. As the median voter has ideal quantity m, this represents a majority of problem writers.
Otherwise, suppose $m < b$. Similar similar reasoning gives that all problem writers with ideal quantity $q \leq m$ will vote for a.

The median voter theorem fails when there are more than two potential outcomes. For instance, suppose there were five problem writers with ideal boba numbers of $1, 2, 3, 4, 5$. Suppose there were four outcomes, $1, 2, 3, 4$. Then, the problem writer with ideal bobas of 1 would choose the first outcome, the problem writer with ideal bobas of 2 would choose the second outcome, the problem writer with ideal bobas of 3 would choose the third outcome, and the problem writers with ideal bobas of 4 and 5 would choose the final outcome. As such, ordering 4 bobas wins despite the median voter being the problem writer whose ideal boba count is 3.

Problem 3.7 (4 pts)**.** *Construct an example of the median voter theorem failing when there are only three outcomes.*

> Suppose there are seven problem writers. Two have ideal bobas of 1, two have ideal bobas of 2, and three have ideal bobas of 3. The three outcomes are $1, 2, 3$. Then, outcome 3 wins but the median problem writer has an ideal boba count of 2. Any example that works receives full credit.

Back in Problem 3.5, we saw that a proposal of purchasing 10 bobas will always get more votes than any other proposal. Suppose Erick still wants an outcome of 7 bobas, but Vicktor, realizing that he previously lost the vote and wanting to take advantage of the median voter theorem, now shifts his proposal to purchasing 10 bobas.

Problem 3.8 (4 pts)**.** *Erick has a friend, Isaack, who can suggest another proposal. Can Isaack suggest another boba amount to make Erick's proposal of* 7 *win? If so, what should Isaack suggest, and why does it cause Erick to win? If not, prove why no proposal works.*

> Yes. Suppose Isaack suggests purchasing 15 bobas. Then, problem writers with ideal boba quantities from 0 to 8 vote for Erick, problem writers with ideal boba quantities from 9 to 12 vote for Vicktor, and problem writers with ideal boba quantities from 13 to 20 vote for Isaack. As such, Erick gets 9 votes, Vicktor gets 4 votes, and Isaack gets 8 votes. Any number that works receives full credit.

Problem 3.9 (4 pts)**.** *Suppose that Erick's proposal is* 15 *bobas and Vicktor's proposal is* 14 *bobas. In this case, can Isaack suggest another boba amount to make Erick's proposal win? If so, what should Isaack suggest, and why does it cause Erick to win? If not, prove why no proposal works.*

> No. No problem writer with ideal boba quantity less than 15 will ever vote for Erick, as Vicktor will always be a better choice. As such, at best Erick wins all problem writers with optimal amounts from 15 to 20, which is 6 votes. However, this means that Vicktor and Isaack will split the remaining 15 votes, and at least one of them will have more than 7 votes. Thus, Erick can never win in this case.

These preceding problems demonstrate how many voting systems might be susceptible to manipulation, often in unexpected ways. For another example, reconsider the preferences of Alice, Bob, and Carl, choosing among outcomes x, y, z:

- Alice: $x \succ y \succ z$;
- Bob: $z \succ x \succ y$;
- Carl: $y \succ z \succ x$.

After realizing Condorcet doesn't produce a result, suppose that the friends ask Dave to run the following:

1. Dave chooses some order of x, y, z;
2. Set the first outcome in Dave's ordering to be the "standing best";
3. Going down Dave's ordering, if any outcome has more people preferring it to the standing best, it becomes the new standing best. Otherwise, it is discarded and the standing best does not change.
4. The outcome that's the standing best at the end is the chosen outcome.

We will call this process pairwise majority rule. For instance, if Dave's ordering is x, y, z then the algorithm proceeds as follows:

1. As x is the first item in Dave's ordering, it starts off as the standing best.

2. The next item in Dave's ordering is y. As Alice and Bob both have $x \succ y$, we have that y is discarded and x is still the standing best.
3. The next item in Dave's ordering is z. As Bob and Carl both have $z \succ x$, we have that z is the new standing best.
4. As all outcomes in Dave's ordering have been considered, the final standing best of z is the final outcome.

Problem 3.10 (2 pts)**.** *Find an ordering for Dave to make x the final outcome.*

We have that x beats y which beats z, so an ordering of z, y, x leads to x being the final outcome. y, z, x also works.

Problem 3.11 (2 pts)**.** *Find an ordering for Dave to make y the final outcome.*

We have that y beats z which beats x, so an ordering of x, z, y leads to y being the final outcome. z, x, y also works.

Problem 3.12 (8 pts)**.** *Show that there is a Condorcet winner if and only if all orderings lead to the same outcome.*

Suppose there is a Condorcet winner, and let it be x. As x is a Condorcet winner, more people prefer x to any other outcome. Thus, when it is x's turn to be compared to the standing best, x becomes the standing best, and will continue to be the standing best until the end.
Conversely, suppose there is no Condorcet winner. Let y be the outcome with respect to some ordering. Since there is no Condorcet winner, there exists some z such that more people prefer z than y. Consider an ordering that starts with $y, z, \ldots$ with the rest being arbitrary. Under pairwise majority rule, the standing best starts with y. In the first iteration, y loses to z and z becomes the new standing best. No matter what happens afterwards, y cannot ever become the standing best again, so the outcome in this order must be different.

4 Arrow's Impossibility Theorem

This final section will work through an understanding and proof of Arrow's Impossibility Theorem, one of the most celebrated results in social choice theory.

While we have previously been concerned about selecting a single "winner" from a set of outcomes, Arrow's Impossibility Theorem is concerned with social choice functions:

Definition 4.1. *A social choice function is a function that takes in individual preferences and outputs some aggregated preferences.*

Social choice functions care about the ranking over all outcomes, not just what the best outcome is. For example, "majority rule" as discussed previously is not a social choice function since it does not tell us how to rank outcomes after the best. However, we can extend majority rule to rank outcomes by the number of people that have it as their favorite: $a \succ b$ if and only if more people have a as their most preferred outcome than b.

For the sake of simplicity, we will require individual preferences to only have strict preference (so no indifferences) and for outputs of social choice functions to also not have indifferences. Of course, these preferences still must be complete and transitive. Going forward, let I be the set of all individuals in a society with generic element i, and let $\succ_i$ denote the preferences of individual i. Let $\succ$ denote the aggregated preferences that a social choice function outputs.

We want social choice functions to satisfy some natural properties:

Definition 4.2. *A social choice function is efficient if whenever $a \succ_i b$ for all i, then $a \succ b$. If everyone prefers one outcome over another, then society also prefers that outcome over the other.*

Definition 4.3. *A social choice function is neutral if whenever all individuals' preferences between a and b are the same as their preferences between x and y, then the social choice function's preference between a and b are the same as its preference between x and y. That is, if the set of voters that prefers a to b is the same as the set of voters that* *prefers x to y, then society ranks a and b the same way they rank x and y. Mathematically, a social choice function being neutral states that if $\{i : a \succ_i b\} = \{i : x \succ_i y\}$ then $a \succ b$ if and only if $x \succ y$.*

Definition 4.4. *A social choice function has a dictator if aggregated preferences $\succ$ are equal to some individual's preferences $\succ_i$.*

To see these definitions in action, consider the following social choice function: Randomly pick one individual from the society and set aggregated preferences to their preferences. This is *efficient* since if $a \succ_i b$ for all i, it must be that the chosen individual prefers a to b, so society prefers a to b as well. This is *neutral* since if $\{i : a \succ_i b\} = \{i : x \succ_i y\}$ then the chosen individual must either (1) prefer a to b and x to y, in which case society also prefers a to b and x to y, or (2) not prefer a to b and not prefer x to y, in which case society also does not prefer a to b and does not prefer x to y. Finally, this social choice rule has a dictator, since the aggregated preferences, by definition, are equal to some individual's preferences. In fact, this social choice function is often called random dictatorship.

Problem 4.1 (10 pts, 2 per part)**.** *Consider the unanimity social choice function: $a \succ b$ if and only if all individuals i have $a \succ_i b$.*

- *Show that this social choice function does not always output complete aggregated preferences.*
- *Show that this social choice function always outputs transitive aggregated preferences.*
- *Show that this social choice function is efficient.*
- *Show that this social choice function is neutral.*
- *Show that there is not always a dictator under this social choice function.*

Not always complete: If there are two outcomes a, b where neither is unanimously better than the other, then neither $a \succ b$ nor $b \succ a$.
Is always transitive: If $a \succ b$ and $b \succ c$, then $a \succ_i b$ and $b \succ_i c$ for all individuals. By transitivity of individual preferences, $a \succ_i c$ for all i. Thus, $a \succ c$.
It is efficient. If $a \succ_i b$ for all i, then a is unanimously preferred to b so $a \succ b$.
It is neutral. Suppose $\{i : a \succ_i b\} = \{i : x \succ_i y\}$. If $\{i : a \succ_i b\} = I$, then $\{i : x \succ_i y\} = I$ as well, and $a \succ b$ and $x \succ y$. Otherwise, if $\{i : a \succ_i b\} \neq I$, then $\{i : x \succ_i y\} \neq I$ as well, so neither $a \succ b$ nor $x \succ y$ hold.
There will not always be a dictator. Any example where the outcome is not complete is an example where there is no dictator as all individual preferences are complete.

Problem 4.2 (15 pts, 3 per part)**.** *Consider the Borda Count social choice function: For each outcome x and individual i, let $x_i = |\{y : y \succ_i x\}|$ and assign x a score of $s(x) = \sum_{i \in I} x_i$. This is the total number of outcomes that are better than x summed over all individuals. Then, $x \succ z$ if and only if $s(x) < s(z)$ (suppose preferences are so that there are never ties).*

- *Show that Borda Count always outputs complete aggregated preferences.*
- *Show that Borda Count always outputs transitive aggregated preferences.*
- *Show that Borda Count is efficient.*
- *Show that Borda Count is not always neutral (an example of inputs where Borda Count is not neutral is sufficient).*
- *Show that there is not always a dictator in Borda Count.*

This is complete. Given any two outcomes a, b either $s(a) < s(b)$ or $s(b) < s(a)$ since there are no ties, so either $a \succ b$ or $b \succ a$.
This is transitive. If $a \succ b$ and $b \succ c$ then $s(a) < s(b)$ and $s(b) < s(c)$ so $s(a) < s(c)$ and thus $a \succ c$.
It is efficient. Suppose $a \succ_i b$ for all i. Then, $a_i < b_i$ for all i. Thus,

$$s(a) = \sum_{i \in I} a_i < \sum_{i \in I} b_i = s(b)$$

so $a \succ b$.
It is not neutral. Suppose there are two individuals with preferences

$$a \succ_1 b \succ_1 x \succ_1 c \succ_1 y \succ_1 z$$

and

$$c \succ_2 b \succ_2 x \succ_2 y \succ_2 z \succ_2 a.$$

Then, $s(a) = 0 + 5 = 5, s(b) = 1 + 1 = 2, s(c) = 3 + 0 = 3$ so $b \succ c$ and $b \succ a$. However,

$$\{i : a \succ_i b\} = \{i : b \succ_i c\} = \{1\}$$

so if neutrality were true, $a \succ b$ if and only if $b \succ c$, which is violated.
There will not always be a dictator. Suppose there once again were two individuals with preferences

$$a \succ_1 b \succ_1 c$$

and

$$c \succ_2 a \succ_2 b.$$

Then, $s(a) = 0 + 1 = 1, s(b) = 1 + 2 = 3, s(c) = 2 + 0 = 2$ so Borda Count gives $a \succ c \succ b$. This is neither individual's preferences.

Can these properties of efficiency, neutrality, and no dictator be simultaneously satisfied? If there is one individual in society, the answer is no.

Problem 4.3 (6 points). *Suppose there is one individual and at least three outcomes. Show that a social choice rule is efficient if and only if it is dictatorial.*

Suppose the social choice rule is dictatorial. Let $\succ_1$ be the one individual's preferences. Then, $\succ = \succ_1$. If $a \succ_i b$ for all i, then this is the same as $a \succ_1 b$, so $a \succ b$.
Conversely, suppose the social choice rule is not dictatorial. Then, there must exist a, b such that $a \succ_1 b$ but $b \succ a$. However, this violates efficiency, as if $a \succ_1 b$ then all individuals have $a \succ_i b$ and the aggregated preferences must have $a \succ b$ as well.

Things are trickier when there are more individuals. To move towards Arrow's Impossibility Theorem, we will impose some additional assumptions on individual preferences. We say that preferences are sufficiently diverse, if there exists outcomes a, b, c such that $a \succ_i c$ for all i, but opinions are split as to where b lies: Some individuals have $a \succ_i b$ while others have $b \succ_i a$; some individuals have $c \succ_i b$ while others have $b \succ_i c$.

For the remainder of the section, suppose that the social choice functions under consideration are efficient and neutral, and that preferences are sufficiently diverse.

Problem 4.4 (6 pts). *Suppose there are two individuals (with sufficiently diverse preferences). Show that there exists outcomes x and y such that one individual prefers x to y, the other individual prefers y to x, and the social choice function outputs $x \succ y$.*

As preferences are sufficiently diverse, there exist outcomes x, y, z such that $x \succ_1 z$ and $x \succ_2 z$. Without loss of generality, suppose that $x \succ_1 y$ and $y \succ_2 x$. By efficiency, it must be that $x \succ z$.
If the social choice function outputs $x \succ y$, then $x \succ_1 y, y \succ_2 x, x \succ y$ as desired.
If the social choice function outputs $y \succ x$, then $y \succ_2 x, x \succ_1 y, y \succ x$ as desired by swapping x and y in the problem statement.

Problem 4.5 (8 pts). *Again, suppose there are two individuals. Further suppose there exists outcomes x and y such that $x \succ_1 y, y \succ_2 x, x \succ y$. Show that for any other outcomes a, b if $a \succ_1 b$ then the social choice function outputs $a \succ b$.*

Consider person two's preferences. If $a \succ_2 b$, then both individuals prefer a to b so $a \succ b$ by efficiency. If $b \succ_2 a$, then

$$\{i : a \succ_i b\} = \{1\} = \{i : x \succ_i y\}$$

so the social choice function being neutral gives that $a \succ b$ if and only if $x \succ y$. As $x \succ y$ is given, we must have $a \succ b$.

Problem 4.6 (4 pts). *Suppose there are two individuals and there exists outcomes x and y such that $x \succ_1 y, y \succ_2 x, x \succ y$. Show that individual 1 is a dictator.*

We will show that $\succ$ is equivalent to $\succ_i$. By the preceding problem, for any other outcomes a, b if $a \succ_1 b$ then $a \succ b$. Repeatedly applying this going down individual one's preference ordering gives the desired result.

Problem 4.7 (6 pts). *Use the preceding three problems to prove Arrow's Impossibility Theorem for two individuals: If preferences are sufficiently diverse, then any efficient and neutral social function must have a dictator.*

As preferences are sufficiently rich and the social choice function is efficient, Problem 4.4 gives that there exists outcomes x and y such that one individual prefers x to y, the other individual prefers y to x, and the social choice function outputs $x \succ y$. Let that individual be individual 1. With this setup, the hypotheses of Problem 4.6 are satisfied so individual 1 is a dictator.

Arrow's Impossibility theorem states there is no efficient, neutral and non-dictatorial social choice function for n individuals, $n \geq 1$. We have proven the cases $n = 1$ and $n = 2$. The general case, if this power round has sparked your interest, will prove to be a good read!