Preliminaries

Time Limit: 80 minutes.

Maximum Score: 122

Instructions: Problems that use the words "prove," "show," or "why" require an explanation or proof. Otherwise, an answer (without explanation or proof) is sufficient. Answers should be written on sheets of scratch paper, clearly labeled, with every problem on its own sheet. If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. Problem 3.2 1/2, Problem 3.2 2/2).

Indicate your team ID number on each piece of paper that you submit. Only submit one set of solutions for the team. Do not turn in any scratch work. In your solution for a given problem, you may cite the statements of earlier problems (but not later ones) without additional justification, even if you haven't solved them. The problems are ordered by content, NOT DIFFICULTY. It is to your advantage to attempt problems from throughout the test. While completing the round, you should not consult the internet or any materials outside of the content of this test (including results not covered in this power round). You may not use calculators.

1 What is Social Choice?

Everyone has things that they want. You might like apples, while your SMT teammates might like bananas. How should you decide who gets what items?

One approach is through prices: At a grocery store, those who are willing to pay the listed price for apples will purchase apples, and those who are willing to pay the listed price for bananas will purchase bananas. Yet in many cases, prices don't work too well.

For instance, if we're trying to decide who should next receive a kidney donation or who the next President should be, using a market-based mechanism would attract much criticism. In these situations, the science of how to aggregate people's diverse wants becomes important. This Power Round will introduce some of the mathematics behind how societies make decisions.

2 Preferences

Preferences can be represented by a *binary relationship*, which we will eventually develop into a *partial order*. A natural order that you are probably familiar with is "$\geq$" over the real numbers. For example, consider a set G of all items in a grocery store. You might like apples more than oranges or be indifferent between the two fruits, which can be represented as "apple $\succeq$ orange". If this is the case, we say that apples are weakly preferred to oranges. By convention, everything is weakly preferred to itself, so "apple $\succeq$ apple". If oranges are also weakly preferred to apples, then "orange $\succeq$ apple" as well. In this case, you would be *indifferent* between apples and oranges and like them the same. In general, if $x \succeq y$ and $y \succeq x$, we can write that $x \sim y$.

Problem 2.1 (1pt). *Is indifference a symmetric relationship? That is, if $x \sim y$, then is it necessarily the case that $y \sim x$?*

What happens if apples are weakly preferred to oranges, but oranges are *not* weakly preferred to apples, so "apple $\succeq$ orange" but "orange $\not\succeq$ apple"? In this case, there is no longer indifference between apples and oranges, and we denote "apple $\succ$ orange". In general, if $x \succeq y$ and $y \not\succeq x$, we can write that $x \succ y$ and say that x is *strictly* preferred to y.

Without any additional assumptions, binary relationships are not too interesting. For example, someone can walk into a grocery store and not have any thoughts about what they like at all. To rule such trivial cases out, a common assumption is that binary relationships are *complete*.

Definition 2.1. *A binary relation $\succeq$ over a set G is complete if for all elements x and y in G, either $x \succeq y$ or $y \succeq x$ (or both).*

Similarly, another common assumption is transitivity: If apples are better than oranges, and oranges are better than bananas, then apples should be better than oranges.

Definition 2.2. *A binary relation $\succeq$ over a set G is transitive if $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.*

Problem 2.2 (6 pts). *Suppose $\succeq$ over a finite set G is complete and transitive. Show that G has a maximal element: There exists some x in G such that for any y in G, we have that $x \succeq y$.*

Problem 2.3 (2 pts). *Give an example of a complete and transitive binary relation over an infinite set such that no maximal element exists.*

Problem 2.4 (4 pts). *Suppose $\succeq$ is a transitive binary relation. Is $\succ$ a transitive binary relation? Prove or provide a counterexample.*

2.1 Utility Function Representations

In general, it is very difficult to work with partial orders directly when defining what someone's preferences are. For instance, if a set has n elements in it, requiring completeness requires at least $\binom{n}{2} + n = \frac{n(n+1)}{2}$ comparisons, a number that grows quickly.

Problem 2.5 (2 pts). *What is the maximum number of weak preferences a partial order (that may not necessarily be complete or transitive) may have over n elements?*

Instead, a useful representation of preferences is through utility functions. Suppose your preferences over $\{x, y, z\}$ was $x \succeq y, y \succeq z, x \succeq z$. One way to represent this is by defining

$$u(x) = 3, u(y) = 2, u(z) = 1$$

and to say that $a \succeq b$ if and only if $u(a) \geq u(b)$.

Definition 2.3 (Utility Function)**.** *Given preferences $\succeq$ over the set G, a utility function u from G to real numbers represents $\succeq$ if for any a, b in G, we have that $a \succeq b$ if and only if $u(a) \geq u(b)$.*

A useful property is that if preferences can be represented by a utility function, natural properties are automatically satisfied. For instance:

Problem 2.6 (4 pts)**.** *Suppose $\succeq$ on G can be represented by the utility function u. Show that $\succeq$ is complete and transitive.*

In general, there can be many utility functions that represent the same preferences.

Definition 2.4. *A function f is monotonic if $x > y$ implies $f(x) > f(y)$.*

Problem 2.7 (4 pts)**.** *Suppose the utility function g represents preferences $\succeq$. Show that if f monotonic, then $f \circ g$ also represents preferences $\succeq$, where $f \circ g$ is the function f composed with g: $(f \circ g)(x) = f(g(x))$.*

3 Voting Systems

An important yet difficult problem plaguing society for years is that of preference aggregation: Given a set of individual preferences over some set of different outcomes, how can we aggregate that into some democratic societal ranking over outcomes?

Suppose you and your teammates want to celebrate surviving the Stanford Math Tournament power round. There are many choices that need to be made: Where should you go? When should the celebration take place?

In general, societies have always struggled with preference aggregation. Even very natural aggregation schemes fail to preserve the desired properties of (individual) preference orderings. Suppose there are three individuals, Alice, Bob, Carl, and three possible outcomes, x, y, z. To simplify notation, we write preferences in the notation of *linear orders*: We list outcomes in a way such that the first outcome is the most-preferred, the second outcome is the second most-preferred, and so on. Linear orders induce complete and transitive preferences. For instance, let Alice, Bob, and Carl's preferences be:

- Alice: $x \succ y \succ z$;
- Bob: $z \succ x \succ y$;
- Carl: $y \succ z \succ x$.

Suppose preferences are aggregated by the following rule: Society has $a \succ b$ if and only if more individuals have $a \succ b$ than $b \succ a$. So for instance, society has $x \succ y$ since Alice and Bob have $x \succ y$, but only Carl has $y \succ x$.

Problem 3.1 (2 pts)**.** *Show that with these individual preferences, the given aggregation rule fails transitivity.*

Similarly, it could be the case that a reasonable aggregation scheme fails to produce an outcome at all.

Definition 3.1. *An outcome x is a Condorcet winner if more individuals have $x \succeq y$ than $y \succeq x$ for all y.*

A Condorcet winner does not always exist. Consider the preferences Alice, Bob, and Carl's preferences from before. None of the outcomes are Condorcet winners:

- Alice and Bob have $x \succ y$ so y cannot be a Condorcet winner;
- Alice and Carl have $y \succ z$ so z cannot be a Condorcet winner;
- Bob and Carl have $z \succ x$ so x cannot be a Condorcet winner.

Problem 3.2 (2 pts)**.** *Construct another example of preferences for which no Condorcet winner exists, and show why no Condorcet winner exists.*

Problem 3.3 (4 points). *Show that if the set of possible outcomes is finite, then either a Condorcet winner exists or there is a sequence of outcomes $a_1, ..., a_n$ such that weakly more people prefer a_i to a_{i+1} for every $i = 1, 2, ..., n-1$ and more people prefer a_n to a_1 (in other words, there is a cycle).*

Yet despite (or perhaps, because of) these difficulties, there can still be a rich theory of how societies aggregate preferences.

For instance, a classic result is the median voter theorem. Suppose an odd number of SMT problem writers are trying to decide on the number of boba teas they should buy at the next problem writing meeting. Each problem writer has an ideal number of bobas, and strictly prefers b bobas to b' bobas if and only if b is closer to their ideal number of bobas than b'. If there are ties, the voter will prefer the higher number of bobas. For instance, a problem writer with an ideal number of 10 bobas will have preferences over $\{0, 1, ..., 20\}$ of:

$$10 \succ 11 \succ 9 \succ 12 \succ 8 \succ 13 \succ 7 \succ 14 \succ 6 \succ 15 \succ 5 \succ 16 \succ 4 \succ 17 \succ 3 \succ 18 \succ 2 \succ 19 \succ 1 \succ 20 \succ 0.$$

Problem 3.4 (2 pts). *Suppose there are 21 problem writers, one for each ideal number of bobas from* 0 *to* 20*. Erick proposes purchasing* 7 *bobas and Vicktor proposes purchasing* 18 *bobas. How many problem writers vote for each proposal, and which proposal wins the majority of votes?*

Problem 3.5 (4 pts). *Once again, suppose there are 21 problem writers with one at each ideal number of bobas from* 0 *to* 20*. Show that a proposal of purchasing* 10 *bobas will always get more votes than any other proposal.*

This constitutes the reasoning behind the median voter theorem. Suppose there is any odd number of problem writers, and each of them has an arbitrary ideal number of bobas. (In particular, each problem writer's ideal number of bobas no longer has to be between 0 and 20, but they still prefer numbers closer to their ideal over numbers further from their ideal.) Define the median problem writer to be the problem writer with the median ideal number of bobas out of all problem writers. The median voter theorem says that, given two proposed number of bobas, the number that the median problem writer likes more will win a majority of votes.

Problem 3.6 (6 pts). *Prove this version of the median voter theorem.*

The median voter theorem fails when there are more than two potential outcomes. For instance, suppose there were five problem writers with ideal boba numbers of $1, 2, 3, 4, 5$. Suppose there were four outcomes, $1, 2, 3, 4$. Then, the problem writer with ideal bobas of 1 would choose the first outcome, the problem writer with ideal bobas of 2 would choose the second outcome, the problem writer with ideal bobas of 3 would choose the third outcome, and the problem writers with ideal bobas of 4 and 5 would choose the final outcome. As such, ordering 4 bobas wins despite the median voter being the problem writer whose ideal boba count is 3.

Problem 3.7 (4 pts). *Construct an example of the median voter theorem failing when there are only three outcomes.*

Back in Problem 3.5, we saw that a proposal of purchasing 10 bobas will always get more votes than any other proposal. Suppose Erick still wants an outcome of 7 bobas, but Vicktor, realizing that he previously lost the vote and wanting to take advantage of the median voter theorem, now shifts his proposal to purchasing 10 bobas.

Problem 3.8 (4 pts). *Erick has a friend, Isaack, who can suggest another proposal. Can Isaack suggest another boba amount to make Erick's proposal of 7 win? If so, what should Isaack suggest, and why does it cause Erick to win? If not, prove why no proposal works.*

Problem 3.9 (4 pts). *Suppose that Erick's proposal is* 15 *bobas and Vicktor's proposal is* 14 *bobas. In this case, can Isaack suggest another boba amount to make Erick's proposal win? If so, what should Isaack suggest, and why does it cause Erick to win? If not, prove why no proposal works.*

These preceding problems demonstrate how many voting systems might be susceptible to manipulation, often in unexpected ways. For another example, reconsider the preferences of Alice, Bob, and Carl, choosing among outcomes x, y, z:

- Alice: $x \succ y \succ z$;
- Bob: $z \succ x \succ y$;
- Carl: $y \succ z \succ x$.

After realizing Condorcet doesn't produce a result, suppose that the friends ask Dave to run the following:

1. Dave chooses some order of x, y, z;

2. Set the first outcome in Dave's ordering to be the "standing best";
3. Going down Dave's ordering, if any outcome has more people preferring it to the standing best, it becomes the new standing best. Otherwise, it is discarded and the standing best does not change.
4. The outcome that's the standing best at the end is the chosen outcome.

We will call this process pairwise majority rule. For instance, if Dave's ordering is x, y, z then the algorithm proceeds as follows:

1. As x is the first item in Dave's ordering, it starts off as the standing best.
2. The next item in Dave's ordering is y. As Alice and Bob both have $x \succ y$, we have that y is discarded and x is still the standing best.
3. The next item in Dave's ordering is z. As Bob and Carl both have $z \succ x$, we have that z is the new standing best.
4. As all outcomes in Dave's ordering have been considered, the final standing best of z is the final outcome.

Problem 3.10 (2 pts)**.** *Find an ordering for Dave to make x the final outcome.*

Problem 3.11 (2 pts)**.** *Find an ordering for Dave to make y the final outcome.*

Problem 3.12 (8 pts)**.** *Show that there is a Condorcet winner if and only if all orderings lead to the same outcome.*

4 Arrow's Impossibility Theorem

This final section will work through an understanding and proof of Arrow's Impossibility Theorem, one of the most celebrated results in social choice theory.

While we have previously been concerned about selecting a single "winner" from a set of outcomes, Arrow's Impossibility Theorem is concerned with social choice functions:

Definition 4.1. *A social choice function is a function that takes in individual preferences and outputs some aggregated preferences.*

Social choice functions care about the ranking over all outcomes, not just what the best outcome is. For example, "majority rule" as discussed previously is not a social choice function since it does not tell us how to rank outcomes after the best. However, we can extend majority rule to rank outcomes by the number of people that have it as their favorite: $a \succ b$ if and only if more people have a as their most preferred outcome than b.

For the sake of simplicity, we will require individual preferences to only have strict preference (so no indifferences) and for outputs of social choice functions to also not have indifferences. Of course, these preferences still must be complete and transitive. Going forward, let I be the set of all individuals in a society with generic element i, and let $\succ_i$ denote the preferences of individual i. Let $\succ$ denote the aggregated preferences that a social choice function outputs.

We want social choice functions to satisfy some natural properties:

Definition 4.2. *A social choice function is efficient if whenever $a \succ_i b$ for all i, then $a \succ b$. If everyone prefers one outcome over another, then society also prefers that outcome over the other.*

Definition 4.3. *A social choice function is neutral if whenever all individuals' preferences between a and b are the same as their preferences between x and y, then the social choice function's preference between a and b are the same as its preference between x and y. That is, if the set of voters that prefers a to b is the same as the set of voters that prefers x to y, then society ranks a and b the same way they rank x and y. Mathematically, a social choice function being neutral states that if $\{i : a \succ_i b\} = \{i : x \succ_i y\}$ then $a \succ b$ if and only if $x \succ y$.*

Definition 4.4. *A social choice function has a dictator if aggregated preferences $\succ$ are equal to some individual's preferences $\succ_i$.*

To see these definitions in action, consider the following social choice function: Randomly pick one individual from the society and set aggregated preferences to their preferences. This is *efficient* since if $a \succ_i b$ for all i, it must be that the chosen individual prefers a to b, so society prefers a to b as well. This is *neutral* since if $\{i : a \succ_i b\} = \{i : x \succ_i y\}$ then the chosen individual must either (1) prefer a to b and x to y, in which case society also prefers a to b and x

to y, or (2) not prefer a to b and not prefer x to y, in which case society also does not prefer a to b and does not prefer x to y. Finally, this social choice rule has a dictator, since the aggregated preferences, by definition, are equal to some individual's preferences. In fact, this social choice function is often called random dictatorship.

Problem 4.1 (10 pts, 2 per part). *Consider the unanimity social choice function: $a \succ b$ if and only if all individuals i have $a \succ_i b$.*

- *Show that this social choice function does not always output complete aggregated preferences.*
- *Show that this social choice function always outputs complete aggregated preferences.*
- *Show that this social choice function is efficient.*
- *Show that this social choice function is neutral.*
- *Show that there is not always a dictator under this social choice function.*

Problem 4.2 (15 pts, 3 per part). *Consider the Borda Count social choice function: For each outcome x and individual i, let $x_i = |\{y : y \succ_i x\}|$ and assign x a score of $s(x) = \sum_{i \in I} x_i$. This is the total number of outcomes that are better than x summed over all individuals. Then, $x \succ z$ if and only if $s(x) < s(z)$ (suppose preferences are so that there are never ties).*

- *Show that Borda Count always outputs complete aggregated preferences.*
- *Show that Borda Count always outputs transitive aggregated preferences.*
- *Show that Borda Count is efficient.*
- *Show that Borda Count is not neutral.*
- *Show that there is not always a dictator in Borda Count.*

Can these properties of efficiency, neutrality, and no dictator be simultaneously satisfied? If there is one individual in society, the answer is no.

Problem 4.3 (6 points). *Suppose there is one individual and at least three outcomes. Show that a social choice rule is efficient if and only if it is dictatorial.*

Things are trickier when there are more individuals. To move towards Arrow's Impossibility Theorem, we will impose some additional assumptions on individual preferences. We say that preferences are sufficiently diverse, if there exists outcomes a, b, c such that $a \succ_i c$ for all i, but opinions are split as to where b lies: Some individuals have $a \succ_i b$ while others have $b \succ_i a$; some individuals have $c \succ_i b$ while others have $b \succ_i c$.

For the remainder of the section, suppose that the social choice functions under consideration are efficient and neutral, and that preferences are sufficiently diverse.

Problem 4.4 (6 pts). *Suppose there are two individuals (with sufficiently diverse preferences). Show that there exists outcomes x and y such that one individual prefers x to y, the other individual prefers y to x, and the social choice function outputs $x \succ y$.*

Problem 4.5 (8 pts). *Again, suppose there are two individuals. Further suppose there exists outcomes x and y such that $x \succ_1 y, y \succ_2 x, x \succ y$. Show that for any other outcomes a, b if $a \succ_1 b$ then the social choice function outputs $a \succ b$.*

Problem 4.6 (4 pts). *Suppose there exists outcomes x and y such that $x \succ_1 y, y \succ_2 x, x \succ y$. Show that individual 1 is a dictator.*

Problem 4.7 (6 pts). *Use the preceding three problems to prove Arrow's Impossibility Theorem for two individuals: If preferences are sufficiently diverse, then any efficient and neutral social function must have a dictator.*

Arrow's Impossibility theorem states there is no efficient, neutral and non-dictatorial social choice function for n individuals, $n \geq 1$. We have proven the cases $n = 1$ and $n = 2$. The general case, if this power round has sparked your interest, will prove to be a good read!