1. *Viktor wants to build a big sandcastle with a triangle base.*

   What is the maximum area of a right triangle with hypotenuse 10?

   Solution: Let the two other sides be a, b. From the Pythagorean Theorem, we know $a^2 + b^2 = 10^2$. The area of the triangle is $\frac{ab}{2}$.

   By the AM-GM inequality, we have

$$\frac{a^2 + b^2}{2} \geq \sqrt{a^2 b^2} = ab \implies \frac{a^2 + b^2}{4} \geq \frac{ab}{2}.$$

   Therefore the area is at most $\frac{10^2}{4} = \boxed{25}$. This is achieved by an isosceles right triangle.

2. For his 21st birthday, Arpit would like to play a game of 21. He would like to achieve 21 total points by drawing three cards and adding up their point values, with the third card's point value being worth twice as much (multiplied by two in the sum). If there are infinite cards with point values 1 through 14, how many ways are there for him to get to 21? Note that the order of the cards drawn matters.

   Solution: We do casework on the point value of the third card. If the third card's point value is z, then $21 - 2z$ points are left for the other two cards. The number of ways to split $21 - 2z$ points among two cards is $20 - 2z$ (a card's point value must be at least 1), giving us a total of

$$\sum_{z=1}^{9} 20 - 2z = 90$$

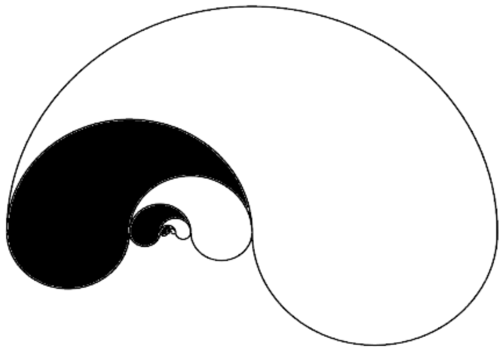
   ways to split the points. However, since the point values only go up to 14, we need to subtract some possibilities for the cases in which $21 - 2z = 19$ or 17. When $21 - 2z = 19$, we cannot allow $1 + 18, 2 + 17, 3 + 16$, and $4 + 15$, which gives 8 possibilities (multiply by 2 to account for ordering). When $21 - 2z = 17$, we cannot allow $1 + 16$ and $2 + 15$, which gives 4 possibilities. Subtracting these gives us an answer of $90 - 8 - 4 = \boxed{78}$.

3. Compute the number of positive integers n less than 100 such that n^2 divides $n!$.

   Solution: First, if n is prime it is easy to see that n^2 cannot divide $n!$. Thus n must be composite. If n is not a perfect square, it is equal to ab with $a, b < n$ and $a \neq b$. n, a, b all occur in the expansion of $n!$, so n satisfies the statement. If $n = m^2$ is a perfect square, when $m > 2$ we have $2m < n$. Thus all perfect squares except 4 are also valid.

   There are 25 primes less than 100, so the answer is $99 - 25 - 1 = \boxed{73}$.

4. The image to the right is comprised of black and white interlocking shapes that are similar to each other. Each shape's perimeter is composed of one "outer" semicircular arc and two smaller "inner" semicircular arcs. The largest shape, which is white, has an outer radius of length 1 and an inner radius of length 1/2. If the pattern depicted continues infinitely, what is the positive difference between the total area of the white shapes and the total area of the black shapes?



Solution: Note that the area of each shape is the same as the area of a semicircle with the same radius as the outer radius of the shape. The sum of the white areas is

$$\frac{\pi}{2}(2^0 + 2^{-4} + 2^{-8} + ...).$$

The sum of the black areas is

$$\frac{\pi}{2}(2^{-2} + 2^{-6} + ...) = \frac{1}{4} \cdot \frac{\pi}{2}(2^0 + 2^{-4} + 2^{-8} + ...).$$

Subtracting, we have

$$\frac{\pi}{2}(2^0 + 2^{-4} + 2^{-8} + ...) - \frac{1}{4} \cdot \frac{\pi}{2}(2^0 + 2^{-4} + 2^{-8} + ...) = \frac{3}{4} \cdot \frac{\pi}{2}(2^0 + 2^{-4} + 2^{-8} + ...)$$

$$= \frac{3\pi}{8}\left(\frac{1}{1 - \frac{1}{16}}\right)$$

$$= \frac{3\pi}{8} \cdot \frac{16}{15}$$

$$= \boxed{\tfrac{2}{5}\pi}.$$

5. *Dean is at the beach making sandcastles too, but there's a problem — he's ambidextrous! His sandcastles always end up looking the same from the left and right.*

   What is the largest 4-digit palindrome that can be written as a sum of three 3-digit palindromes?

   Solution: We can write the condition as the equation

$$\overline{abba} = \overline{cdc} + \overline{efe} + \overline{ghg}.$$

   Notice that $a \leq 2$ since $\overline{cdc} + \overline{efe} + \overline{ghg} \leq 999 \times 3 = 2997$. Since we are trying to find the largest such palindrome, we let $a = 2$. Hence we can write

$$2002 + 110b = 101(c + e + g) + 10(d + f + h).$$

Since the left hand side ends in a 2, the right hand side must as well. Therefore, $(c+e+g)$ ends in a 2. This sum can be 12 or 22 (since $1 \leq e, f, g, \leq 9$). We let $c+e+g=22$ as otherwise it is not possible to achieve a sum greater than 2000. Now we have $2002+110b = 2222+10(d+f+h)$, which gives $11b-(d+f+h)=22$.

Since $d+f+h \leq 9+9+9=27$ we can write $11b \leq 49$, which gives us $b \leq 4$.

Thus, the largest possible palindrome is $\boxed{2442}$. One possibility for how it can be written as a sum of three 3-digit palindromes is $2442 = 999+999+444$.

6. *Misha is the sandcastle building god.*

   2024 Greek gods and goddesses (numbered as 1 to 2024 from most to least important) are coming together for a banquet. You are deity number 2024. There are 2024 seats labelled with the deities' numbers, and the gods enter in order from least to most important. When you enter, you choose a random seat to sit in (which may be your designated seat). When god i enters, if their seat is empty they sit in it. Otherwise (if you are in it), they pay you $i+1$ prayers to move out of it so that they can sit. You then take an unoccupied seat at random. Compute the expected total number of prayers you earn through this procedure.

   Solution: Note that when god i enters, the probability that you are in god i's seat is $\frac{1}{i+1}$, for each i. This is because when god i enters, there are $i+1$ remaining seats (including your own) that have not already been occupied by another god, and your selection process has an equal probability of choosing any of these $i+1$ seats. Therefore, by linearity of expectation the answer is $\sum_{i=1}^{2023} \frac{i+1}{i+1} = \boxed{2023}$.

7. *Kat told Viktor that equilateral triangles make better sandcastles.*

   In equilateral triangle ABC, points D, E, and F are chosen on line segments $\overline{BC}, \overline{CA}$, and $\overline{AB}$ such that $\angle FCA = 2\angle EBC = 4\angle DAB$. Line AD meets $\overline{CF}$ at X and $\overline{BE}$ at Y. Given that the four points C, E, X, Y are concyclic, compute $\angle FCA$.

   Solution: Without a loss of generality, assume the intersection of CF and BE is to the right of AD. Let $\angle FCA = x$, we will compute x. Since C, E, X, Y are concyclic, it follows that

   $$\angle DAB + \angle EBA = \angle EYX = x.$$

   We are given $\angle DAB = x/4$, and can compute $\angle EBA = \angle ABC - \angle EBC = 60^\circ - x/2$. Thus $x/4 + 60^\circ - x/2 = x$, which upon solving gives $x = \boxed{48^\circ}$.

8. How many positive integers n are there such that the following equation has at least one real solution in x?

   $$x^4 + 4x^3 + 24x^2 + 40x + n = 0$$

   Solution: Using that $(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$, and that $(x+1)^2 = x^2 + 2x + 1$, we have that:

   $$\begin{aligned} x^4 + 4x^3 + 24x^2 + 40x + n &= (x+1)^4 + 18x^2 + 36x + n - 1 \\ &= (x+1)^4 + 18(x+1)^2 + n - 19. \end{aligned}$$

Then since $(x+1)^4 + 18(x+1)^2 \geq 0$, it follows that if $n \leq 19$, the expression has a solution. Since n must be a positive integer, we see that the answer is $\boxed{19}$.

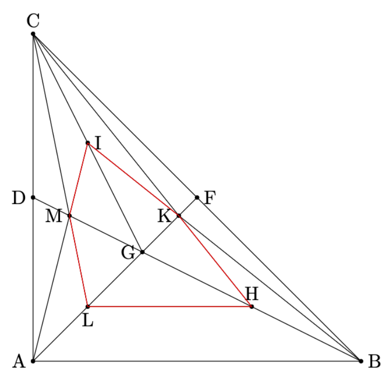
9. *Eric comes and destroys all the sandcastles. He gives builders this problem instead:*

   Given that $3^{36} + 3^{25} + 3^{13} + 1$ has three prime factors, compute its largest prime factor.

   Solution: Note that $3^{36} + 3^{25} + 3^{13} + 1 = (3^{12}+1)^3$. We then have $3^{12} + 1 = (3^4+1)(3^8 - 3^4 + 1)$. Since $3^4 + 1 = 2 \cdot 41$ has two prime factors, the largest prime factor is then $3^8 - 3^4 + 1 = \boxed{6481}$.

10. Consider the triangle ABC where $AC = 1$ and $AB = 1$. G is the centroid of ABC. Points D, F, L, H, and I are the midpoints of AC, BC, AG, GB, and CG respectively. Let M be the point where CL intersects BD and let K be the point where CH intersects AF. Compute the ratio of the area of pentagon $IMLHK$ to the area of triangle $\triangle ABC$.

    Solution:



The area of ABC is $(\frac{1}{2})(1)(1)) = \frac{1}{2}$. Draw a new triangle ILH. Notice that because each of the points forming the triangle are the midpoints of their respective lines, the triangle ILH is similar. In particular, it is twice as small. Thus, the area of ILH is $(\frac{1}{2})(\frac{1}{2})(\frac{1}{2}) = \frac{1}{8}$. Now, we want to calculate the area of the remaining smaller triangles IML and IKH. Notice that the area of these triangles are the same because they are congruent, so we only need to find one. Let's find IML. Recognise that we know that $IL = 0.5$. We want to find the distance from IL to M, i.e. the height of the triangle, in order to calculate its area. Recall that because G is the centroid, both I and L sit one third along the way of their respective median lines. Thus, we can claim the the perpendicular distance from AC to IL is $\frac{1}{3}$ the length of the total "height". Using a similar argument, notice that M lies $\frac{2}{3}$ the perpendicular distance from the line AC to IL. So, now we can claim that the perpendicular distance from IL to M is $\frac{1}{3}$ the length of the perpendicular distance from AC to IL, i.e. $(\frac{1}{3})(\frac{1}{6}) = \frac{1}{18}$

Now we can calculate the area of IML to be $(\frac{1}{2})(\frac{1}{2})(\frac{1}{18}) = \frac{1}{72}$, which we double to count the triangle IKH to get $\frac{1}{36}$. Finally, to calculate our ratio we will have

$$\frac{\frac{1}{8} + \frac{1}{36}}{\frac{1}{2}} = \boxed{\frac{11}{36}}.$$

11. Let T be a triangle with the largest possible area whose vertices all have coordinates of the form (p, q) such that p, q are prime numbers less than 100. How many lattice points are either contained in T or lie on the boundary of T?

Solution: Observe that the area of the triangle will be at most half of the area of the smallest rectangle which contains all the points of the triangle on its edges and whose sides are parallel to the axes of the grid. Suppose the triangle has vertices at (p_1, q_1), (p_2, q_2), (p_3, q_3). Let $P = \{p_1, p_2, p_3\}$ and let $Q = \{q_1, q_2, q_3\}$. Then, the area of the rectangle is

$$(\max(P) - \min(P))(\max(Q) - \min(Q)).$$

If we set the triangle to be a right triangle, then it attains exactly half of the area of the bounding rectangle. Therefore, it suffices to maximize this value. Then, clearly the values we should choose are $\max(P) = \max(Q) = 97$, $\min(P) = \min(Q) = 2$, and so the vertices of the triangle T are either $(2, 2), (2, 97), (97, 97)$ or $(2, 2), (97, 2), (97, 97)$.

Then, since this is a right triangle, it contains at least half of the points in the rectangle, including the points on the boundary. Without loss of generality, we choose T to have vertices $(2, 2), (97, 2), (97, 97)$. Then, considering that the slope of the line along the hypotenuse is 1, all points (x, y) in T or on the boundary of T satisfy

$$2 \le x \le y \le 97.$$

Therefore, the number of points is exactly

$$\begin{aligned}\sum_{y=2}^{97}\sum_{x=2}^{y} 1 = \sum_{y=2}^{97}(y-1) = \sum_{y=1}^{96} y \\ &= \frac{96(96+1)}{2} \\ &= 48 \cdot 97 \\ &= \boxed{4656}.\end{aligned}$$

12. What is the smallest positive integer with the property that the sum of its proper divisors is at least twice as great as itself? (The proper divisors of a number are the positive divisors of the number excluding the number itself.)

Solution: Consider the sum-of-divisors" function, $\sigma(n)$. When $n = p_1^{e_1} \cdots p_m^{e_m}$ (its prime factorization) , $\sigma(n) = \prod_{k=1}^{n} \frac{p_k^{e_k+1}-1}{p_k-1}$. In order for the sum of the proper divisors to be at least $2n$, $\sigma(n)$ must be at least $3n$, i.e. we must have

$$\prod_{k=1}^{n} \frac{p_k^{e_k+1} - 1}{p_k - 1} \ge 3 \prod_{k=1}^{n} p_k^{e_k},$$

or equivalently,

$$\prod_{k=1}^{n} \left(1 + \frac{1 - p_k^{-e_k}}{p_k - 1}\right) \ge 3.$$

Note that $\prod_{k=1}^{n}\left(1+\frac{1-p_k^{-e_k}}{p_k-1}\right)$ is bounded above by $\prod_{k=1}^{n}\left(1+\frac{1}{p_k-1}\right)$. Therefore, a necessary (but not sufficient) condition is for $\prod_{k=1}^{n}\left(1+\frac{1}{p_k-1}\right) \geq 3$ to hold. Experimenting, we see that when n only has prime divisors $2, 3$, there is equality. To account for the effect of omitting $p_k^{-e_k}$, n must at least have a third prime divisor, preferably 5, since 5 contributes the least to the size of n, but the most to the inequality.

Now it's a matter of trial and error. If n contains a fourth prime divisor, n would be at least 210. If we can find a minimal value for n less than 210 with only prime divisors $2, 3, 5$ with the property desired, that n would be the answer. Trial and error shows that $30, 60$ and 90 do not work, but 120 does. So $\boxed{120}$ is the answer desired.

13. Compute the remainder when $(10!)^{20}$ is divided by 2024.

Solution: We have $2024 = 2^3 \cdot 11 \cdot 23$. It is clear that $(10!)^{20} \equiv 0 \pmod 8$. By Wilson's theorem, we know that $10! \equiv -1 \pmod{11}$, so $(10!)^{20} \equiv 1 \pmod{11}$. Now, note that

$$22! \equiv (10!) \cdot 11 \cdot 12 \cdot (-10) \cdot (-9) \cdots (-1) \equiv (10!)^2 \cdot 11 \cdot 12 \pmod{23}.$$

Again by Wilson's theorem, we have $(10!)^2 \cdot 11 \cdot 12 \equiv -1 \pmod{23}$. Note that $(-2) \cdot 11 \equiv -22 \equiv 1 \pmod{23}$ and $2 \cdot 12 \equiv 1 \pmod{23}$. Then, multiplying by $(-2) \cdot 2$ on both sides of the equivalence gives us $(10!)^2 \equiv 4 \pmod{23}$. We know that $(10!)^{22} \equiv 1 \pmod{23}$ by Fermat's Little Theorem, so $(10!)^{20} \equiv 4^{-1} \equiv 6 \pmod{23}$.

Let $(10!)^{20} = 11k + 1$ for some integer k. Then, $11k + 1 \equiv 6 \pmod{23}$, and multiplying both sides by -2 gives us $k \equiv 13 \pmod{23}$. Then, let $k = 23l + 13$ for some integer l, so $(10!)^{20} = 253l + 144$. In order for this number to be divisible by 8, l must be divisble by 8. Since $8 \cdot 253 = 2024$, the remainder we seek is $\boxed{144}$.

14. A right square pyramid with height 12 and a base of side length 10 is inscribed in sphere S. Compute the largest possible radius of a sphere that lies inside S and is tangent to one of the lateral faces of the pyramid.

Solution: We first find the radius of S, which we denote r. Note that the triangle with vertices at the center of S, the center of the base of the pyramid, and one of the vertices of the base of the pyramid is a right triangle with legs of lengths $5\sqrt{2}$ and $12 - r$, and a hypotenuse of length r. Using the Pythagorean theorem, we have $(5\sqrt{2})^2 + (12 - r)^2 = r^2$, which gives us $r = \frac{97}{12}$. We next compute the circumradius of one of the lateral faces of the pyramid. The height of one of the lateral faces is $\sqrt{5^2 + 12^2} = 13$ and the base is 10, so its area is 65. The lateral face has two sides of length $\sqrt{(5\sqrt{2})^2 + 12^2} = \sqrt{194}$ and a third side of length 10. The circumradius then is $\frac{(\sqrt{194})(\sqrt{194})(10)}{(4)(65)} = \frac{97}{13}$.

Now, note that the circumcircle of a lateral side of the pyramid cuts S into two pieces. Our answer is the radius of the sphere that is tangent to S, tangent to the lateral side at its circumcenter, and lies in the larger piece of S. To compute the distance d from the circumcenter of the lateral side to the center of S, note that the center of S, the circumcenter of the lateral side, and a point on the circumcircle of the lateral side form a right triangle with legs of lengths d and $\frac{97}{13}$, and hypotenuse of length $\frac{97}{12}$. We have $d = 97\sqrt{(1/12)^2 - (1/13)^2} = \frac{97 \cdot 5}{156}$. The radius that we want is then $\frac{d+r}{2} = 97 \cdot \frac{\frac{5}{156} + \frac{1}{12}}{2} = 97 \cdot \frac{3}{52} = \boxed{\frac{291}{52}}$.

15. For any integer $n \geq 2$, with prime factorization $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, we let $f(n) = \sum_{i=1}^{k} p_i a_i$. For example, since $90 = 2^1 \cdot 3^2 \cdot 5^1$, we have $f(90) = 2 \cdot 1 + 3 \cdot 2 + 5 \cdot 1 = 13$. Let m be the minimum value that $f(n)$ can take on for all integers $n > 2024$. Find the smallest integer $k \geq 2$ such that $f(k) = m$.

Solution: We first prove that the minimum value of $f(n)$ for $n > 2024$ is achieved by a value of n that only has prime factors 2 and 3. If n has a prime factor of 5, each factor of 5 can be replaced with $2 \cdot 3 = 6$ so that the value $f\left(n \cdot \frac{6}{5}\right) = f(n)$ remains the same since $2 + 3 = 5$, while $n \cdot \frac{6}{5} > n$ so if $n > 2024$ then $n \cdot \frac{6}{5} > 2024$.

If n has a prime factor of 7, note that $2^3 > 7^1$, and $2 \cdot 3 < 7 \cdot 1$, so $f\left(n \cdot \frac{2^3}{7}\right) < f(n)$ while $n \cdot \frac{2^3}{7} > n > 2024$. For any odd number d greater than 7, we see that if $2^{(d-3)/2} > d - 2$, then $2^{(d-1)/2} > d$. Using this observation inductively, we can see that any odd prime factor p can be replaced with $2^{(p-1)/2}$ so that $2^{(p-1)/2} > p$ while $2(p-1) < p$. This proves our claim.

We find that for $n = 2^{11} = 2048 > 2024$, $f(n) = 2 \cdot 11 = 22$, and for $n = 3^7 = 2187 > 2024$, $f(n) = 3 \cdot 7 = 21$. We now prove that $f(n) = 21$ is the minimum for $n > 2024$. To get lower than 21, we need $n = 2^a 3^b$ so that $f(n) = 2a + 3b \leq 20$. Since $3^2 > 2^3$, the maximum possible value of $2^a 3^b$ given that $2a + 3b \leq 20$ is $2 \cdot 3^6 = 1458 < 2024$. Thus, it is not possible to achieve a value of $f(n)$ lower than 21 for $n > 2024$.

We find that because 21 is not prime, we need k to be composite. If $k = 38$, then $f(k) = 19 + 2 = 21$, and it is clear that for other $a + b = 21$, $k = ab > 38$. If k has three or more (not necessarily distinct) prime factors, this would also result in $n > 38$. This is because for any integers $a, b \geq 2$, we have $ab - (a + b) = (a-1)(b-1) - 1 \geq 0$, so we always achieve a lower product by using less factors to add up to 21. Thus, our answer is $k = \boxed{38}$.

16. Compute $\left\lfloor \frac{2023}{4202} \right\rfloor + \left\lfloor \frac{2023 \cdot 2}{4202} \right\rfloor + \dots + \left\lfloor \frac{2023 \cdot 4201}{4202} \right\rfloor$.

Solution: Since $\gcd(2023, 4202) = 1$, we see that $2023, 2 \cdot 2023, \dots (4202 - 1) \cdot 2023$ all have different residues modulo 4202, as otherwise there would be some difference $2023(k_1 - k_2) \equiv 0 \pmod{4202}$ where $k_1, k_2 \in \{1, 2, \dots, 4201\}$ are distinct, but this is not possible. So, each residue from 1 to 4201 is represented exactly once.

This means that if we ignore the floors and add the fractions in the given expression directly, we then need to subtract $\frac{1+2+\dots+4201}{4202}$ to get the desired answer. We have

$$\left\lfloor\frac{2023}{4202}\right\rfloor + \left\lfloor\frac{2023\cdot 2}{4202}\right\rfloor + \ldots + \left\lfloor\frac{2023\cdot 4201}{4202}\right\rfloor = \frac{(2023 + 2023\cdot 2 + \ldots 2023\cdot 4201) - (1 + 2 + \ldots + 4201)}{4202}$$
$$= \frac{(2023-1)(\frac{4202\cdot 4201}{2})}{4202}$$
$$= \frac{2022\cdot 4201}{2} = 1011\cdot 4201$$
$$= \boxed{4247211}.$$

17. Triangle ABC has side lengths $AB = 24$ and $AC = 22$, and the radius of its circumcircle is 13. Compute the sum of the possible lengths of BC.

Solution: We first place points A and B on a circle of radius of 13 such that 24. There are then possible positions for point C. We first consider the case in which $\triangle ABC$ is acute. Let AD be a diameter of the circle. Since $\triangle ABD$ and $\triangle ACD$ are right triangles, we find that $BD = \sqrt{26^2 - 24^2} = 10$ and $CD = \sqrt{26^2 - 22^2} = 8\sqrt{3}$. By Ptolemy's theorem on $ABDC$, we have $26(BC) = 22\cdot 10 + 24\cdot 8\sqrt{3}$, which gives us $BC = \frac{110+96\sqrt{3}}{13}$. In the case that $\triangle ABC$ is obtuse, we again use Ptolemy's theorem to find that $26(BC) + 22\cdot 10 = 24\cdot 8\sqrt{3}$, which gives us $BC = \frac{-110+96\sqrt{3}}{13}$. The sum of the possible lengths of BC is $\frac{110+96\sqrt{3}}{13} + \frac{-110+96\sqrt{3}}{13} = \boxed{\frac{192\sqrt{3}}{13}}$.

18. Consider the following rule for moves on the two-dimensional integer lattice: for each coordinate (b, c) that you are on, move to $(b+1, c)$ if $0 = x^2 + bx + c$ has no real solution, and move to $(b, c+1)$ otherwise. If you begin at $(0, 0)$, what coordinates do you land on after 2024 moves?

Solution: Note that we move to $(b+1, c)$ if $b^2 < 4c$ and $(b, c+1)$ otherwise. In other words, we move to $(b, c+1)$ whenever b is sufficiently large and are forced'' to move to $(b+1, c)$ when needed to satisfy $b^2 \geq 4c$. Then, we see that the number of moves required to reach $(n, n^2/4)$ for even n is $n + n^2/4$. We can find n such that this expression is approximately 2024. In fact, solving $n^2/4 + n = 2024$ gives us that n is exactly 88, so the coordinates we land on are $(88, 88^2/4) = \boxed{(88, 1936)}$.

19. What is the largest composite number n such that the sum of the digits of n is larger than the greatest divisor of n, excluding n itself?

Solution: The greatest divisor of n is not less than $\sqrt{n}$. If $n \geq 10^4$, then the greatest divisor is at least 100. The maximum possible digit sum for a five-digit number is $5\cdot 9 = 45$, so n must have less than five digits. For four-digit numbers, the maximal sum of digits is $4\cdot 9 = 36$. We see that $36^2 = 1296$, so we must have $n \leq 1296$, but then the digit sum of n is at most $1+2+9+9 = 21$.

Therefore, the largest such number is at most three digits. We proceed to solve this by iteratively reducing the upper bound of such a number, which in turn reduces the maximum possible sum of digits, which in turn reduces the upper bound of such a number.

For three digit numbers, the upper bound of the sum of digits is $9+9+9 = 27$, so the largest such number is less than $27^2 = 729$. But then, the leading digit can be at most 7, so the maximal digit sum is now $7+9+9 = 25$, so we can lower the bound to $25^2 = 625$. Similarly, now the leading digit is 6, so by the same logic, the upper bound is at most $24^2 = 576$. The digit sum is now bounded by $\max(5+7+9, 4+9+9) = 22$, which gives us an upper bound of $22^2 = 484$. This upper bounds the digit sum at $\max(4+8+9, 3+9+9) = 21$, and the upper bound for n is reduced to $21^2 = 441$. We now see that if the first digit is 4, the digit sum is no more than $4+4+9 = 17$, so the first digit must be at most 3. Then, the greatest divisor is at most 19. Now, we check that $19^2 = 361$ and $18^2 = 324$ do not work, but $17^2 = 289$ does, so the answer must be $\boxed{289}$.

20. Consider cutting the ellipse $y^2 + \frac{x^2}{9} = 1$ by the line $y = \sqrt{7}x + 4$. What is the largest area bounded by the ellipse and the line?

Solution: Let's solve this for general radius r and $y = mx + b$. To do this, consider stretching the $(1,1)$ circle: that is, the circle represented by $x^2 + y^2 = 1$. Then, stretching by a factor of r in the x direction will yield the $(r, 1)$ ellipse. In other words, we began with an (x, y) coordinate system and end up with a $(x/r, y)$ system. Therefore, we can instead consider cutting the circle $x^2 + y^2 = 1$ with the line $y = mr \cdot x + b$, and then multiply the resulting area by r to get the correct ellipse area (the way to show this is to technically compute the determinant of the Jacobian of this transformation).

Although we may rotate this to simplify calculation, we can also calculate it directly. Substitution yields $(m^2r^2 + 1) \cdot x^2 + (2mbr) \cdot x + b^2 - 1 = 0$, so $x = \frac{-mbr \pm \sqrt{m^2r^2+1-b^2}}{1+m^2r^2}$ and $y = mr \cdot \frac{-mbr \pm \sqrt{m^2r^2+1-b^2}}{1+m^2r^2} + b$. Then, the length of the chord between these two solutions is

$$\sqrt{\left(\frac{2\sqrt{m^2r^2+1-b^2}}{m^2r^2+1}\right)^2 + m^2r^2 \cdot \left(\frac{2\sqrt{m^2r^2+1-b^2}}{m^2r^2+1}\right)^2} = 2\sqrt{1 - \frac{b^2}{1+m^2r^2}}.$$

Then, Law of Cosines on the triangle whose side lengths are the chord and two radii of the circle yields that the subtended angle is

$$\cos\theta = \frac{\frac{4b^2}{1+m^2r^2} - 2}{2} = \frac{2b^2}{1+m^2r^2} - 1$$

so $\sin\theta = \frac{2b\sqrt{m^2r^2+1-b^2}}{1+m^2r^2}$.

Now, notice that the cut off area is $\pi \cdot \frac{\theta}{2\pi} - \frac{\sin\theta}{2} = \frac{1}{2}(\theta - \sin\theta)$. Hence, our final ellipse area is

$$\frac{r(2\pi - \theta + \sin\theta)}{2},$$

where $\sin\theta$ is as above.

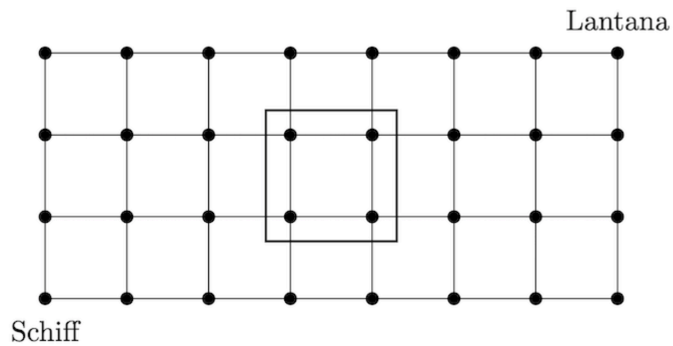
Let $r = 3, m = \sqrt{7}, b = 4$. Then $\sin\theta = \frac{\sqrt{3}}{2}$. The largest possible value for θ is $\theta = \frac{2\pi}{3}$ and our final answer is

$$\frac{3}{2}\left(\frac{4\pi}{3} + \frac{\sqrt{3}}{2}\right) = \boxed{2\pi + \frac{3\sqrt{3}}{4}}.$$

21. Consider the 4 by 8 grid of points below that represents the Stanford campus. Stanford has developed a way to teleport Main Quad (represented by the rectangle on the lattice) anywhere on campus such that 4 points are contained within the rectangle and none of these points are dorms

(the dimensions of Main Quad remain the same as in the figure). The bottom-left and top-right corners of the grid are dorms.

Abby wants to bike from Schiff to Lantana, but does not want to pass through any points in Main Quad. If Abby only moves from one lattice point to another in the up and right directions (and Main Quad does not move as she bikes), compute the sum of the number of paths she can take for all possible positions of Main Quad.



Solution: We consider the grid as having 3 rows and 7 columns. We assign coordinates to the lattice points so that the bottom-left corner is $(0,0)$ and the top-right corner is $(7,3)$. Let us first consider the case in which Main Quad is in the top row. Denote the column of Main Quad as $c \in \{1,2,\ldots,6\}$ ($c \neq 7$ since Main Quad cannot contain Lantana). Now, either $(c+1,0)$ or $(c+1,1)$ must be the first point Abby reaches with x-coordinate $c+1$. The number of paths for which $(c+1,0)$ is the first is $\binom{c+1}{c+1}\binom{6-c+3}{3} = \binom{9-c}{3}$. The number of paths for which $(c+1,1)$ is the first is $\left(\binom{c+2}{1} - \binom{c+1}{c+1}\right)\binom{6-c+2}{2} = (c+1)\binom{8-c}{2}$. We have the sum

$$\begin{aligned}\sum_{c=1}^{6}\left(\binom{9-c}{3} + (c+1)\binom{8-c}{2}\right) &= \left(\sum_{i=3}^{8}\binom{i}{3}\right) + \left(\sum_{i=2}^{7}\binom{i}{2}\right) + \left(\sum_{i=2}^{7}\sum_{j=2}^{i}\binom{j}{2}\right)\\ &= \binom{9}{4} + \binom{8}{3} + \sum_{i=2}^{7}\binom{i+1}{3}\\ &= 2\binom{9}{4} + \binom{8}{3},\end{aligned}$$

by the Hockey-Stick Identity.

By symmetry, the sum for positions of Main Quad in the bottom row is the same. We now consider if Main Quad is in the middle row. We denote the column of Main Quad as $c \in \{1,\ldots,7\}$. Now Abby can either pass through the point $(c+1,0)$ or $(c-2,3)$ (we ignore negative coordinates—they indicate that there is no such path). For $(c+1,0)$, the number of paths is $\binom{9-c}{3}$ (same as one of the cases for Main Quad in the top row). For $(c-2,3)$, the number of paths is $\binom{c-2+3}{3} = \binom{c+1}{3}$. We let $\binom{a}{b} = 0$ if $a < b$. We have the sum

$$\sum_{c=1}^{7}\binom{9-c}{3} + \binom{c+1}{3} = \binom{9}{4} + \binom{9}{4}$$

again by the Hockey-Stick Identity.

Our final sum is $2\left(2\binom{9}{4}+\binom{8}{3}\right)+\binom{9}{4}+\binom{9}{4}=6\binom{9}{4}+2\binom{8}{3}=\boxed{868}$.

22. *Note: this round consists of a cycle, where each answer is the input into the next problem.*

Let $\mathcal{A}$ be the answer to problem 24. Let ABC be a triangle with area $\mathcal{A}$ and $\angle ABC = 90^\circ$. Let M and N be the midpoints of BC and BA, respectively. Let P be a point on the circumcircle of ABC such that arc APC is distinct from arc ABC, and let P' be the point of intersection of line PM and the circumcircle of $\triangle ABC$ such that $P \neq P'$. Let T be the intersection of PA and $P'B$. If T also lies on line MN and $\tan\angle BAC = 3$, compute the area of $\triangle ABT$.

Solution: Note that $MN \parallel CA$, so $\angle MTP = \angle CAP = \angle CBP = \angle MBP$. This means that points B, M, P, T are cyclic. Let $\angle MBP' = \theta$. Note that $\angle APC = 90^\circ$, so $\angle TPM = 90^\circ - \angle P'PC = 90^\circ - \angle P'BC = 90^\circ - \theta$. Also, $\angle TBM = 180^\circ - \theta$ and $\angle TBM + \angle TPM = 180^\circ$, so we get $90^\circ - \theta + 180^\circ - \theta = 180^\circ$, which gives us $\theta = 45^\circ$. Now, let the foot of the perpendicular from T to line BC be T'. We see that $\triangle TT'B$ is a $45-45-90$ triangle, so $TT' = T'B = x$. Let the length of segment BC be a. We have that $\triangle TT'M$ is a right triangle with $\angle TMT' = \angle ACB$, so $\tan\angle TMT' = \tan\angle ACB = \frac{1}{3}$. Thus, $\frac{1}{3} = \frac{TT\prime}{T\prime B+BM} = \frac{x}{x+a/2}$, and we get $x = \frac{a}{4}$. We also have $BT = x\sqrt{2} = \frac{a\sqrt{2}}{4}$.

The area of $\triangle ABT$ is $\frac{1}{2}(BA)(BT)\sin(45^\circ) = \frac{1}{2}(BA)\frac{a\sqrt{2}}{4}\frac{1}{\sqrt{2}} = \frac{1}{2}\frac{(BA)(BC)}{4} = \frac{[ABC]}{4} = \boxed{\frac{\mathcal{A}}{4}}$.

23. Let R be the answer to problem 22 and set $N = 100 + R$. Let $f : \{2, 3, ..., N\} \to \{2, 3, ..., N\}$ be any function such that there are exactly $N + 15$ ordered pair solutions to the equation $f(x) - f(y) = 0$. Suppose F is the collection of these functions f which maximize

$$\log_2 f(2)\log_3 f(3)\cdots\log_N f(N).$$

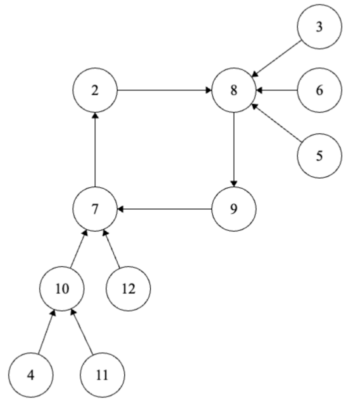
Over all $f \in F$, compute the maximum possible value of $\sum_{i=2}^{N} f(i) - i$.

Solution: Let m_i be the number of values x such that $f(x) = i$, for each i from 2 through N. Then, we have the constraints that $\sum_{i=2}^{N} m_i = N - 1$ and $\sum_{i=2}^{N} m_i^2 = N + 15$.

Then, we have (by subtracting and dividing by 2) $\sum_{i=2}^{N}\binom{m_i}{2} = 8$, so at most 8 indices i satisfy $m_i > 1$. We can characterize them exactly:

$$(2:8), (2:5, 3:1), (2:2, 3:2), (2:2, 4:1).$$

To determine which of these is best, let us rephrase the problem as a graph. The vertices are $2, 3, ..., N$ and the (directed) edges are from x to $f(x)$ (this is also known as a functional graph). Such a graph can *always* be characterized as a collection of disjoint cycles, where each vertex of each cycle possibly has a directed tree hanging off of it. Here is an example:



Next, notice that if we have a cycle $x_1, x_2, \ldots, x_k$, then the value $\log_{x_1}(x_2) \cdot \log_{x_2}(x_3) \cdots \log_{x_k}(x_1) = 1$. Indeed, consider all but the last term: by log rules, this is $\log_{x_1}(x_k)$ and so multiplying by the last term gives 1.

Now, let us determine which of the above four cases is the best for us. We claim that for $N \geq 100$ it is optimal to choose the case of 8 indices m_i with $m_i = 2$ (that is, 8 indices not on the cycle). This is intuitive since there are the most terms, so at the end of this solution we will give a formal proof.

In this case, we claim that the optimum is $\log_2(N) \cdot \log_3(N-1) \cdots \log_9(N-7)$ (the order of the bases does not matter). This is achievable by having $f(2) = N, f(3) = N-1, \ldots, f(9) = N-7$. To show we cannot do better, let us label our non-cycle nodes as $v_1, v_2, \ldots, v_8$. The only way we can possibly do better than the above quantity is to have, say, $f(v_2) = v_1$ (that is, some nodes do not directly point to a cycle). Then, consider any path to the cycle of length > 1: for example, $v_2, v_1, x \in$ cycle. The contribution from this path to the product is $\log_{v_2}(v_1) \cdot \log_{v_1}(x) = \log_{v_2}(x)$, so having a path of length > 1 means that we can have at most 7 log terms in the product. This is shown below to be suboptimal.

Now, notice that every $f \in F$ has the same value of $\sum_{i=2}^{N} f(i) - i$ (cycles do not change the sum): it is

$$\sum_{i=2}^{N} f(i) - i = \sum_{i=2}^{N} i - \sum_{i=2}^{N} i + \sum_{i=2}^{9} (N - (i-2)) - i = 8N + 16 - 2\sum_{i=2}^{9} i = \boxed{8N - 72}.$$

Now let us show the above claim: that $N \geq 100$ suffices for the first case to be optimal. To show this, it suffices to show that

$$\log_2(N) \cdot \log_3(N-1) \cdots \log_9(N-7) \geq \log_2(N) \cdot \log_3(N) \cdots \log_8(N).$$

Note that all other cases have at most 7 logs in the product, so they can never be more than the RHS.

For a fairly simple proof, begin by dividing through both sides by $\ln(2) \cdot \ln(3) \cdots \ln(8)$. From here, we claim that $\ln(N-k) \geq \ln(N) - \frac{k}{N-k}$. Indeed, since ln is a concave function, and the derivative at $N-k$ is $\frac{1}{N-k}$, linearization implies that $\ln(N) \leq \ln(N-k) + \frac{k}{N-k}$.

Therefore, we have that the ratio of the terms is

$$\frac{\log_9(N-7)}{\ln^7(N)}\prod_{i=0}^{6}\ln(N-i) \geq \frac{\log_9(N-7)}{\ln^7(N)}\prod_{i=0}^{6}(\ln(N)-\frac{i}{N-i}) \geq \log_9(N-7)\cdot\left(1-\frac{6}{\ln(N)\cdot(N-6)}\right)^6.$$

When $N \geq 100$, we have that $\log_9(N-7) > 2$. So, we just have to show that the second term is at most $\frac{1}{2}$. The latter term can be very generously approximated by $(\frac{9}{10})^6$ (since $(N-6)\ln N \geq 60$). It is not hard to see that $9^6 = 531441 > 500000$, so this is greater than $\frac{1}{2}$ and the proof is complete.

24. Let M be the answer to problem 23. Compute the number of integers $0 \leq k \leq 2196 - M$ such that $\frac{2196!}{M!k!(2196-M-k)!} \equiv 0 \pmod{13}$.

Solution: Let us rewrite the given format as $\binom{2196}{M}\cdot\binom{2196-M}{k}$. Since $2196 = 13^3 - 1$, $\binom{2196}{M} \not\equiv 0 \bmod 13$ by Lucas's Theorem: indeed, let us write $M = s_0 + 13s_1 + 13^2 s_2$, then $\binom{2196}{M} \equiv \binom{12}{s_0}\binom{12}{s_1}\binom{12}{s_2} \bmod 13$. Each of the latter terms is nonzero and not divisible by 13, implying that $\binom{2196}{M}$ is also not divisible by 13.

Now, we can apply Lucas's theorem once more to the latter term. For this coefficient to be nonzero, every position in k must be majorized by $2196 - M$. In particular, suppose $k = k_0 + 13k_1 + 13^2k_2$. Then, Lucas's Theorem states that $\binom{2196-M}{k} \equiv \binom{12-s_0}{k_0}\binom{12-s_1}{k_1}\binom{12-s_2}{k_2} \bmod 13$.

This implies that the number of k which make this coefficient nonzero is $(13-s_0)(13-s_1)(13-s_2)$ and the answer by complementary counting is $A = 2197 - M - (13-s_0)(13-s_1)(13-s_2)$.

Now let us stitch together the three cycle round problems. If the output of this problem is A, then the input is $M = 8\cdot(\frac{A}{4}+100) - 72 = 2A + 728$.

Note that $728 = 440_{13}$. So, let us take the equation $A = 2197 - 728 - 2A - (13-s_0)(13-s_1)(13-s_2) \bmod 13$. We will also need to write $A = a_0 + 13a_1 + 13^2a_2$ and note that $s_0 \equiv 2a_0 \bmod 13$.

Then, we have that $3A \equiv s_0s_1s_2 \bmod 13$, or in other words using that $3^{-1} = 9 \bmod 13$, $a_0 \equiv 5a_0s_1s_2 \bmod 13$. This implies that either $a_0 = 0$ or $s_1s_2 \equiv 8 \bmod 13$.

Consider the first case. Then, we may write $A = 13B$ and divide the whole equation by 13: $3B = 113 - (13-s_1)(13-s_2)$. Since $B \leq \lfloor\frac{2197-728}{3\cdot 13}\rfloor = 37$, it is in fact possible to check all possible B (speeding up computation by first taking mod 13 to find that $B \equiv 3 - 9s_1s_2 \bmod 13$). Doing so gives one solution: $B = 12$ and thus $A = 12\cdot 13 = 156$.

Now, consider the second case when $s_1s_2 \equiv 8 \bmod 13$ and $a_0 \neq 0$. Then, there are only 12 possibilities for the pair s_1, s_2: they are $(1,8),(2,4),(3,7),(4,2),(5,12),(6,10),(7,3),(8,1),(9,11),(10,6),(11,9),(12,5)$. Substituting $A = \frac{M-728}{2}$ gives us $\frac{M-728}{2} = 2197 - M - (13-s_0)(13-s_1)(13-s_2)$. Letting $M = s_0 + 13s_1 + 13^2s_2$, we can solve for s_0 to get $s_0 = \frac{2s_1s_2-23s_1+13s_2-56}{(2s_1s_2-16)/13-2s_1-2s_2+27}$. Each of the possible pairs s_1, s_2 does not result in a valid value of s_0 (which must be an integer between 0 and 12, inclusive). So, the only possible solution is $A = \boxed{156}$. We then find that the answer to problem 22 is $\frac{156}{4} = 39$ and the answer to problem 23 is $8(100+39) - 72 = 1040$.

25. Frank composes a random 15-note melody where each note is either A, B, or C. What is the probability that no sequence of 5 consecutive notes occurs more than once in the melody he

composes? (For example, in the melody ABCABCABCABCABC the sequence ABCAB occurs more than once.)

For an estimate of E, you will get $\max(0, 25 - \lceil 500 \mid E - X \mid \rceil)$ points, where X is the true answer.

Solution: A possible estimate can be obtained by considering pairs of 5 consecutive notes in the 15-note melody. There are 11 spans of 5 consecutive notes, giving us $\binom{11}{2} = 55$ pairs. If chosen independently, the probability that two spans of 5 consecutive notes are not the same is $\frac{242}{243}$, so we can estimate the probability of no sequence being repeated as $\left(\frac{242}{243}\right)^{55} \approx 1 - \frac{55}{243} \approx 0.774$. We can consider this a lower bound as it considers all pairs spans regardless of whether they are overlapping. For an upper bound, we can consider non-overlapping spans, of which there are 21 pairs. This gives us $\left(\frac{242}{243}\right)^{21} \approx 1 - \frac{21}{243} \approx 0.914$. Averaging the two values gives us 0.844, which gets 19 points.

Python code for the exact answer:

```
def countNoteSequences():
  total_len = 15
  notes = range(3)
  notes_played = []
  seq_five = set()
  result = recurse(notes_played, notes, total_len, seq_five)
  print(result)
  print(result / len(notes) ** total_len)


def recurse(notes_played, notes, total_len, seq_five):
  if len(notes_played) == total_len:
    return 1

  num_seq = 0
  for i in notes:
    notes_played.append(i)

    if len(notes_played) >= 5:
      if tuple(notes_played[-5:]) in seq_five:
        notes_played.pop()
        continue
      else:
        seq_five.add(tuple(notes_played[-5:]))

    num_seq += recurse(notes_played, notes, total_len, seq_five)

    if len(notes_played) >= 5:
      seq_five.remove(tuple(notes_played[-5:]))

    notes_played.pop()

  return num_seq
```

```
if __name__ == `__main__':
  countNoteSequences()
```

26. A prime number is said to be *toothless* if none of its digits are 2, 4, or 8. Estimate the number of *toothless* primes with at most 8 digits.

For an estimate of E, you will get $\max\left(0, 25 - \left\lceil\frac{|E-X|}{4000}\right\rceil\right)$ points, where X is the true answer.

Solution: There are 7^8 numbers that are at most 8 digits and do not contain the digits $2, 4, 8$. A naive estimate would be to use the Prime Number Theorem and assume that each of the 7^8 numbers has an equal probability of being prime to that of a random positive integer with at most 8 digits. This gives us

$$\frac{7^8}{\ln(10^8)} \approx 312953,$$

which gives 0 points. We can find a better estimate by considering that for a random positive integer with at most 8 digits, the probability that the last digit is relatively prime to 10 is $\frac{4}{10}$, while if the integer cannot contain the digits $2, 4$, or 8 the probability is $\frac{4}{7}$. We obtain a new estimate by multiplying by the ratio of these probabilities:

$$\frac{7^8}{\ln(10^8)} \cdot \frac{\frac{4}{7}}{\frac{4}{10}} \approx 447075,$$

which gives 17 points.

Python code for exact answer:

```
import sympy

ans = 0

for i in range(1, 10 ** 8):
    if sympy.isprime(i):
        num = i

        toothless = True
        while num > 0:
            if (num % 10) in {2, 4, 8}:
                toothless = False
                break
            num //= 10

        if toothless:
            ans += 1

print(ans)
```

27. Two binary sequences a, b are uniformly randomly chosen from all binary sequences of length 200. At each step, a random digit of a is flipped, and the digits of b are uniformly randomly permuted. Let X be the expected number of steps before a and b are the same. Estimate $X - 2^{200}$.

For an estimate of E, you will get $\max\left(0, 25 - \left\lceil \frac{|E-S|}{10} \right\rceil\right)$ points, where S is the true value of $X - 2^{200}$.

Solution: Let $n = 200$. Suppose a starts at j ones and b at k ones. Then, we can decompose the expected number of steps as the expected number of steps until a has k ones, and then the expected number of steps until they match from this point.

Let us look at the second step first. It is "well known" (Ehrenfest Urns) that once a is at k ones, it will return there every $\frac{2^n}{\binom{n}{k}}$ steps on average. Since each of the $\binom{n}{k}$ permutations of k ones is equally likely, this means that the expected number of steps until the two match is $\frac{2^n}{\binom{n}{k}} \cdot (\binom{n}{k} - 1) = 2^n - \frac{2^n}{\binom{n}{k}}$ (the -1 is if they match immediately).

Now, let $E(j, k)$ be the expected number of steps to get from j ones to k ones. Our final expected value is thus

$$\sum_{k=0}^{n} \frac{\binom{n}{k}}{2^n}\left(2^n - \frac{2^n}{\binom{n}{k}}\right) + \sum_{k=0}^{n}\sum_{j=0}^{n} \frac{\binom{n}{k}}{2^n}\frac{\binom{n}{j}}{2^n} E(j,k) = 2^n - (n+1) + S$$

It suffices to estimate S. To get a good estimate, note that in expectation we should see j and k being on opposite sides of 100 (if we draw a number line), and around 10 to 30 apart. So, really we can approximate the quantity by instantiating $j = 90, k = 110$. In this range, there is approximately a $\frac{1}{2}$ probability of flipping a 0 to 1 or 1 to 0. Thus, for any $i \in [j, k]$, we can write $E(i, i+1) = 1 + \frac{1}{2}(E(i-1, i) + E(i, i+1))$ (this is saying that we always take a step and then with probability $\frac{1}{2}$ we go back to $i - 1$ ones and have to get from $i - 1$ to i and then from i to $i + 1$ again), which implies that $E(i, i + 1) = 2 + E(i - 1, i)$.

Then, this implies that

$$E(90, 110) = \sum_{i=90}^{109} E(i, i+1) = 20E(89, 90) + \sum_{i=90}^{109} 2(i - 89) = 20E(89, 90) + 420.$$

It remains to estimate $E(89, 90)$. A crude estimate is something like 5 or 10 (using some bounding of probabilities similarly to the above $\frac{1}{2}$ estimate), which give final answers of $520 - 201 = 319$ and $620 - 201 = 419$, respectively. These are enough for 18 and 21 points, respectively.