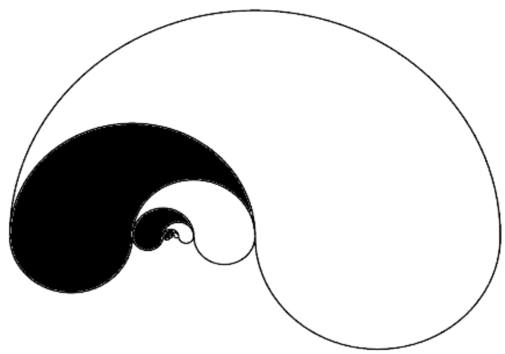
1. *Viktor wants to build a big sandcastle with a triangle base.*

   What is the maximum area of a right triangle with hypotenuse 10?

2. For his 21st birthday, Arpit would like to play a game of 21. He would like to achieve 21 total points by drawing three cards and adding up their point values, with the third card's point value being worth twice as much (multiplied by two in the sum). If there are infinite cards with point values 1 through 14, how many ways are there for him to get to 21? Note that the order of the cards drawn matters.

3. Compute the number of positive integers n less than 100 such that n^2 divides $n!$.

4. The image to the right is comprised of black and white interlocking shapes that are similar to each other. Each shape's perimeter is composed of one "outer" semicircular arc and two smaller "inner" semicircular arcs. The largest shape, which is white, has an outer radius of length 1 and an inner radius of length $1/2$. If the pattern depicted continues infinitely, what is the positive difference between the total area of the white shapes and the total area of the black shapes?



5. *Dean is at the beach making sandcastles too, but there's a problem — he's ambidextrous! His sandcastles always end up looking the same from the left and right.*

   What is the largest 4-digit palindrome that can be written as a sum of three 3-digit palindromes?

6. *Misha is the sandcastle building god.*

   2024 Greek gods and goddesses (numbered as 1 to 2024 from most to least important) are coming together for a banquet. You are deity number 2024. There are 2024 seats labelled with the deities' numbers, and the gods enter in order from least to most important. When you enter, you choose a random seat to sit in (which may be your designated seat). When god i enters, if their seat is empty they sit in it. Otherwise (if you are in it), they pay you $i+1$ prayers to move out of it so that they can sit. You then take an unoccupied seat at random. Compute the expected total number of prayers you earn through this procedure.

7. *Kat told Viktor that equilateral triangles make better sandcastles.*

   In equilateral triangle ABC, points D, E, and F are chosen on line segments $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ such that $\angle FCA = 2\angle EBC = 4\angle DAB$. Line AD meets $\overline{CF}$ at X and $\overline{BE}$ at Y. Given that the four points C, E, X, Y are concyclic, compute $\angle FCA$.

8. How many positive integers n are there such that the following equation has at least one real solution in x?

$$x^4 + 4x^3 + 24x^2 + 40x + n = 0$$

9. *Eric comes and destroys all the sandcastles. He gives builders this problem instead:*

   Given that $3^{36} + 3^{25} + 3^{13} + 1$ has three prime factors, compute its largest prime factor.

10. Consider the triangle ABC where $AC = 1$ and $AB = 1$. G is the centroid of ABC. Points D, F, L, H, and I are the midpoints of AC, BC, AG, GB, and CG respectively. Let M be the point where CL intersects BD and let K be the point where CH intersects AF. Compute the ratio of the area of pentagon $IMLHK$ to the area of triangle $\triangle ABC$.

11. Let T be a triangle with the largest possible area whose vertices all have coordinates of the form (p, q) such that p, q are prime numbers less than 100. How many lattice points are either contained in T or lie on the boundary of T?

12. What is the smallest positive integer with the property that the sum of its proper divisors is at least twice as great as itself? (The proper divisors of a number are the positive divisors of the number excluding the number itself.)

13. Compute the remainder when $(10!)^{20}$ is divided by 2024.

14. A right square pyramid with height 12 and a base of side length 10 is inscribed in sphere S. Compute the largest possible radius of a sphere that lies inside S and is tangent to one of the lateral faces of the pyramid.

15. For any integer $n \geq 2$, with prime factorization $n = p_1^{a_1} p_2^{a_2} \dots p_k^{a_k}$, we let $f(n) = \sum_{i=1}^{k} p_i a_i$. For example, since $90 = 2^1 \cdot 3^2 \cdot 5^1$, we have $f(90) = 2 \cdot 1 + 3 \cdot 2 + 5 \cdot 1 = 13$. Let m be the minimum value that $f(n)$ can take on for all integers $n > 2024$. Find the smallest integer $k \geq 2$ such that $f(k) = m$.

16. Compute $\left\lfloor \frac{2023}{4202} \right\rfloor + \left\lfloor \frac{2023 \cdot 2}{4202} \right\rfloor + \dots + \left\lfloor \frac{2023 \cdot 4201}{4202} \right\rfloor$.

17. Triangle ABC has side lengths $AB = 24$ and $AC = 22$, and the radius of its circumcircle is 13. Compute the sum of the possible lengths of BC.

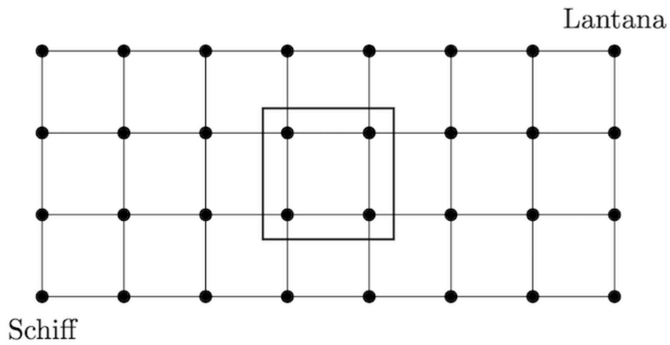
18. Consider the following rule for moves on the two-dimensional integer lattice: for each coordinate (b, c) that you are on, move to $(b + 1, c)$ if $0 = x^2 + bx + c$ has no real solution, and move to $(b, c + 1)$ otherwise. If you begin at $(0, 0)$, what coordinates do you land on after 2024 moves?

19. What is the largest composite number n such that the sum of the digits of n is larger than the greatest divisor of n, excluding n itself?

20. Consider cutting the ellipse $y^2 + \frac{x^2}{9} = 1$ by the line $y = \sqrt{7}x + 4$. What is the largest area bounded by the ellipse and the line?

21. Consider the 4 by 8 grid of points below that represents the Stanford campus. Stanford has developed a way to teleport Main Quad (represented by the rectangle on the lattice) anywhere on campus such that 4 points are contained within the rectangle and none of these points are dorms (the dimensions of Main Quad remain the same as in the figure). The bottom-left and top-right corners of the grid are dorms.

Abby wants to bike from Schiff to Lantana, but does not want to pass through any points in Main Quad. If Abby only moves from one lattice point to another in the up and right directions (and Main Quad does not move as she bikes), compute the sum of the number of paths she can take for all possible positions of Main Quad.



22. *Note: this round consists of a cycle, where each answer is the input into the next problem.*

    Let $\mathcal{A}$ be the answer to problem 24. Let ABC be a triangle with area $\mathcal{A}$ and $\angle ABC = 90^\circ$. Let M and N be the midpoints of BC and BA, respectively. Let P be a point on the circumcircle of ABC such that arc APC is distinct from arc ABC, and let P' be the point of intersection of line PM and the circumcircle of $\triangle ABC$ such that $P \neq P'$. Let T be the intersection of PA and $P'B$. If T also lies on line MN and $\tan \angle BAC = 3$, compute the area of $\triangle ABT$.

23. Let R be the answer to problem 22 and set $N = 100 + R$. Let $f : \{2, 3, ..., N\} \to \{2, 3, ..., N\}$ be any function such that there are exactly $N + 15$ ordered pair solutions to the equation $f(x) - f(y) = 0$. Suppose F is the collection of these functions f which maximize

    $$\log_2 f(2) \log_3 f(3) \cdots \log_N f(N).$$

    Over all $f \in F$, compute the maximum possible value of $\sum_{i=2}^{N} f(i) - i$.

24. Let M be the answer to problem 23. Compute the number of integers $0 \leq k \leq 2196 - M$ such that $\frac{2196!}{M!k!(2196-M-k)!} \equiv 0 \pmod{13}$.

25. Frank composes a random 15-note melody where each note is either A, B, or C. What is the probability that no sequence of 5 consecutive notes occurs more than once in the melody he composes? (For example, in the melody ABCABCABCABCABC the sequence ABCAB occurs more than once.)

    For an estimate of E, you will get $\max(0, 25 - \lceil 500 \mid E - X \mid \rceil)$ points, where X is the true answer.

26. A prime number is said to be *toothless* if none of its digits are 2, 4, or 8. Estimate the number of *toothless* primes with at most 8 digits.

    For an estimate of E, you will get $\max\left(0, 25 - \left\lceil \frac{\mid E-X \mid}{4000} \right\rceil\right)$ points, where X is the true answer.

27. Two binary sequences a, b are uniformly randomly chosen from all binary sequences of length 200. At each step, a random digit of a is flipped, and the digits of b are uniformly randomly permuted. Let X be the expected number of steps before a and b are the same. Estimate $X - 2^{200}$.

For an estimate of E, you will get $\max\left(0, 25 - \left\lceil \frac{|E-S|}{10} \right\rceil\right)$ points, where S is the true value of $X - 2^{200}$.