1. Let $ABCD$ be a square, and let P be a point chosen on segment AC. There is a point X on segment BC such that $PX = PB = 37$ and $BX = 24$. Compute the side length of $ABCD$.

   Solution: Let M be the foot of the altitude from P to BC and let N be the foot of the altitude from P to CD. Since $\triangle BPX$ isosceles with $PB = PX$, we know that M is the midpoint of BX. That is, $BM = 12$, and by the Pythagorean Theorem, we have $PM = \sqrt{37^2 - 12^2} = 35$. Since P lies on AC, by symmetry, we have $PM = PN = MC$, and $BC = BM + MC = BM + PM = 12 + 35 = \boxed{47}$.

2. Let $\triangle ABC$ be an equilateral triangle with side length 6. Three circles of radius 6 are centered at A, B, and C. Compute the radius of the circle that is centered at the center of $\triangle ABC$, is internally tangent to these three circles, and lies in the interior of the three circles.

   Solution: Denote the circle that we want to compute the radius of as ω. Let the center of $\triangle ABC$ be O and let line AO intersect ω at point D such that O lies between A and D. Note that this is also the point where ω is tangent to the circle centered at A. Using $30 - 60 - 90$ triangles, one can show that the circumradius of an equilateral triangle is $\frac{1}{\sqrt{3}}$ of its side length. The desired radius is thus $OD = AD - OA = 6 - \frac{6}{\sqrt{3}} = \boxed{6 - 2\sqrt{3}}$.

3. Let ω be the circle inscribed in regular hexagon $ABCDEF$ with side length 1, and let the midpoint of side BC be M. Segment AM intersects ω at point $P \neq M$. Compute the length of AP.

   Solution: Using the law of cosines on $\triangle ABM$, we have $AM^2 = 1 + \frac{1}{4} - 2 \cdot 1 \cdot \frac{1}{2} \cdot \cos(120^\circ)$, so $AM = \frac{\sqrt{7}}{2}$. Then, let the midpoint of AB be N and note that AN is tangent to ω at N. By Power of a Point, we have

$$(AN)^2 = (AP)(AM)$$

$$\frac{1}{4} = AP \cdot \frac{\sqrt{7}}{2}$$

$$AP = \boxed{\frac{\sqrt{7}}{14}}.$$

4. Let $\triangle OAB$ and $\triangle OA'B'$ be equilateral triangles such that $\angle AOA' = 90^\circ$, $\angle BOB' = 90^\circ$, and $\angle AOB'$ is obtuse. Given that the side length of $\triangle OA'B'$ is 1 and the circumradius of $\triangle OAB'$ is $\sqrt{61}$, compute the side length of $\triangle OAB$.

   Solution: Note that $\angle B'OA = \angle B'OB + \angle BOA = 90^\circ + 60^\circ = 150^\circ$. Thus, by the law of sines, $AB' = 2 \cdot \sqrt{61} \cdot \sin 150^\circ = \sqrt{61}$. Thus by the law of cosines on $\triangle OAB'$, we have $B'A^2 = OA^2 + OB'^2 - 2(OA)(OB') \cos 150^\circ$. Let $x = OA$ be the desired length. Then, substituting lengths, we have $61 = 1 + x^2 + x\sqrt{3}$ or $x^2 + \sqrt{3}x - 60 = 0$. This has roots $\frac{-\sqrt{3} \pm \sqrt{3 + 60 \cdot 4}}{2}$, the only valid root of which is $\boxed{4\sqrt{3}}$.

5. In right triangle $\triangle ABC$ with right angle at B, let I be the incenter and G the centroid. Let the foot of the perpendicular from I to AB be D and the foot of the perpendicular from G to CB be E. Line l is drawn such that l is parallel to DE and passes through B. Line ID meets l at X, and line GE meets l at Y. Given that $AB = 8$ and $CB = 15$, compute the length XY.

Solution: Let DI and EG meet at point F. Note that $DBEF$ is a rectangle. From here, we can see that $\triangle BYE \cong \triangle DEG$ and $\triangle XBD \cong \triangle DEG$. Since $XY \mid\mid DE$, it follows that XY is a dilation by a factor of 2 of DE centered at F. Thus, $XY = 2(DE)$. The hypotenuse of the triangle is 17, so the inradius is $\frac{8+15-17}{2} = 3$. Thus $DB = 3$. Now, let O be the midpoint of AC, and J the foot of the altitude of O to BC. Since $OJ \mid\mid AB$ and O is the midpoint of AC, it follows that J is the midpoint of BC and $BJ = 15/2$. Since G is the centroid of $\triangle ABC$, it follows that $BG : BO = 2 : 3$. Since $GE \mid\mid OJ$, it follows that $BE : BJ = BG : BO = 2 : 3$. That is, $BE = \frac{2}{3} \cdot \frac{15}{2} = 5$. As $DBEF$ is a rectangle, by the Pythagorean Theorem, it follows that $DE = \sqrt{3^2 + 5^2} = \sqrt{34}$. The answer is thus $\boxed{2\sqrt{34}}$.

6. Let $ABCDE$ be a regular pentagon with side length 1. Circles ω_B, ω_C, and ω_D are centered at B, C, and D respectively, each with radius 1. ω_B intersects ω_C inside $ABCDE$ at point F, and ω_C intersects ω_D inside $ABCDE$ at point G. Compute the ratio of the measure of $\angle AFB$ to the measure of $\angle AGB$.

   Solution: Note that $\triangle CDG$ is an equilateral triangle, so $\angle BCG = 108^\circ - 60^\circ = 48^\circ$. Next, we see that $\triangle CBG$ is isosceles with $CG = CB = 1$. Then, $\angle CBG = (180^\circ - 48^\circ)/2 = 66^\circ$. We get $\angle GBA = 108^\circ - 66^\circ = 42^\circ$. Next, we note that AG is the angle bisector of $\angle BAE$ since G is symmetric with respect to points C and D, so $\angle BAG = 54^\circ$. Now we have $\angle AGB = 180^\circ - \angle GBA - \angle BAG = 180^\circ - 42^\circ - 54^\circ = 84^\circ$. Finally, note that $\triangle BAF \cong \triangle CBG$, so $\angle BFA = \angle CGB = 66^\circ$. The ratio we want is then $\frac{66^\circ}{84^\circ} = \boxed{\frac{11}{14}}$.

7. Consider the horizontal line that intersects the ellipse $\frac{x^2}{9} + y^2 = 1$ at points A and B above the x-axis such that $\angle AOB = 120^\circ$, where point O is the origin. Compute the area of the region of the ellipse that lies above this line.

   Solution: Let the midpoint of AB be M and WLOG let A be to the left of the y-axis. Since $\triangle AOM$ is a $30-60-90$ triangle, we know that $A = (-k, k/\sqrt{3})$ for some positive k. Solving for k in $\frac{k^2}{9} + \frac{k^2}{3} = 1$ gives us $k = \frac{3}{2}$.

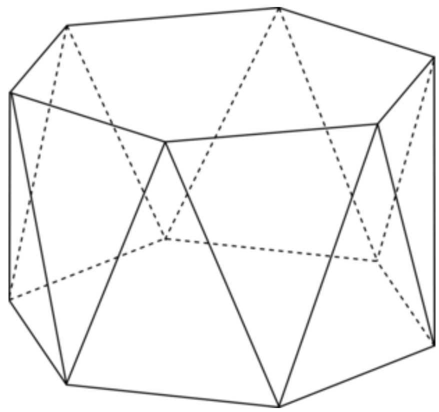
   Consider the circle $x^2 + y^2 = 1$, which can be obtained by compressing the x-coordinates on the ellipse by a factor of 3. Then, $A'M' = \frac{1}{2}$ while we still have $M'O' = \frac{\sqrt{3}}{2}$, so we deduce that $\triangle A'O'M'$ is a $30-60-90$ triangle with $\angle A'O'M' = 30^\circ$. We can then compute the area of the sector subtended by $A'B'$ as $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$. To get the area for the ellipse, we multiply by 3 to get $\boxed{\frac{\pi}{2} - \frac{3\sqrt{3}}{4}}$.

8. Points A and B lie on a circle centered at O such that $AB = 14$. The perpendicular bisector of AB intersects $\odot O$ at point C such that O lies in the interior of $\triangle ABC$ and $AC = 35\sqrt{2}$. Lines BO and AC intersect at point D. Compute the ratio of the area of $\triangle DOC$ to the area of $\triangle DBC$.

   Solution: Let the radius of the circle be r and the midpoint of AB be M. We have $CM = \sqrt{BC^2 - BM^2} = \sqrt{2450 - 49} = 49$ and $OM = \sqrt{BO^2 - BM^2} = \sqrt{r^2 - 49}$. Then, we have $r + \sqrt{r^2 - 49} = 49$, which we can solve to get $r = 25$. Let line BO intersect the circle again at point E. Note that CM and EA are parallel since they are both perpendicular to AB, so $\triangle DOC$ and $\triangle DEA$ are similar since they have the same angles. We also see that $\triangle DEA$ is similar to $\triangle DCB$

since their angles subtend the same arcs of the circle. Then, $\triangle DOC \sim \triangle DCB$ so the ratio of the areas is $\left(\frac{OC}{CB}\right)^2 = \left(\frac{25}{35\sqrt{2}}\right)^2 = \boxed{\frac{25}{98}}$.

9. Consider a prism with regular hexagon bases. We form an antiprism by removing the lateral faces, rotating one of the bases 30° about the axis passing through the centers of the bases, and forming 12 triangular faces between the bases where each triangular face consists of one vertex of one of the bases and the two closest vertices of the other base. Compute the ratio of the volume of the antiprism to the volume of the original prism.



Solution: Consider flattening" the antiprism so that the bases are in the same plane and we have a dodecagon formed from the 12 vertices of the two hexagons. We can compute the volume of a prism with this base and the same height as the original prism and subtract the 12 triangular pyramids that are added on to the antiprism in order to form this new prism. Let the height of the prism be h. The base of the new prism is a regular dodecagon where the distance between its center and a vertex is equal to the side length of the hexagonal base of the original prism. We denote this side length as s. Then, the area of the dodecagonal base is $12 \cdot \frac{1}{2}s^2 \sin(30^\circ) = 3s^2$, so the volume of the new prism is $3s^2h$.

The base of one of the triangular prisms is a triangle with height $s - s\sqrt{3}/2$ and base s. Then, the total volume of the triangular pyramids is $12 \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot (s - s\sqrt{3}/2) \cdot s \cdot h = (2 - \sqrt{3})s^2h$. The volume of the antiprism is $(3 - 2 + \sqrt{3})s^2h = (1 + \sqrt{3})s^2h$. The area of a regular hexagon with side length s is $\frac{3\sqrt{3}}{2}s^2$, so the volume of the original prism is $\frac{3\sqrt{3}}{2}s^2h$. We get that the ratio of the volumes is $\frac{1+\sqrt{3}}{3\sqrt{3}/2} = \boxed{\frac{6+2\sqrt{3}}{9}}$.

10. Given a triangle ABC, let the tangent lines to the circumcircle of $\triangle ABC$ at points A and B intersect at point T. Line CT intersects the circumcircle for a second time at point D. Let the projections of D onto AB, BC, AC be M, N, P respectively. From M draw a line perpendicular to NP intersecting BC at E. If $EC = 5$, $EB = 11$, and $\angle CEP = 120°$, compute the length of CP.

Solution: First note that P, M, N lie on the same line because of Simson's line theorem. We have that quadrilaterals $DNBM$ and $DMPA$ are cyclic, due to right angles formed from the projections of D, and therefore $\angle PDM = \alpha$ and $\angle MDN = \beta$ where $\alpha = \angle BAC$ and $\beta = \angle ABC$. Now let $\angle DCB = \varphi$ which gives us $\angle DCA = \gamma - \varphi$ where $\gamma = \angle ACB$. From the cyclic quadrilaterals, we get $\angle DPM = \angle DAB = \angle DCB = \varphi$ and $\angle DNM = \angle DBM = \angle ACD = \gamma - \varphi$. Using sine law for $\triangle DNP$, we get that

$$\frac{NM}{PM} = \frac{NM}{MD} \cdot \frac{MD}{PM} = \frac{\sin\beta}{\sin(\gamma - \varphi)} \cdot \frac{\sin\varphi}{\sin\alpha}.$$

We now recall that the symmedian of a triangle passes through the intersection of the tangents to the circumcircle at the triangle's vertices, therefore CD is the symmedian for $\triangle ABC$. Since the symmedian is isogonal to the median, we have that $\varphi = \angle BCD = \angle ACS$, where S is the midpoint of AB. Therefore, with sine law for $\triangle ACS$ and $\triangle BCS$, we get

$$\frac{\sin\beta}{\sin(\gamma - \varphi)} \cdot \frac{\sin\varphi}{\sin\alpha} = \frac{CS}{BS} \cdot \frac{AS}{CS} = \frac{AS}{BS} = 1$$

and, therefore, by comparing this with the previous equation, we see that $NM/PM = 1$.

Next, let G be the intersection of PE and the line through B perpendicular to ME and let F be the intersection of EM and CD. Note that EM is the perpendicular bisector of NP, and we have $\angle NEP = 180^\circ - \angle CEP = 60^\circ$, so $\angle MEP = 30^\circ$, $\angle MEN = 30^\circ$, and $\triangle NEP$ is equilateral. Then, $\angle DNM = \angle DNB - \angle MNB = 30^\circ$ and $\angle DCA = \angle DBA = \angle DNM = 30^\circ$. Since $\angle FEP = \angle FCP = 30^\circ$, we know that points C, E, F, P are cyclic. We then have $\angle FPE = \angle FCE$.

We note the following equal angle measures: $\angle BDC = \angle BAC = \angle MAP = \angle MDP$, $\angle DPN = \angle DCN = \angle FPE$. Now, we see that points F and M are isogonal conjugates in quadrilateral $BDPE$. Using law of sines, we have

$$\frac{\sin\angle EBF}{\sin\angle BEF} = \frac{EF}{BF}$$
$$\frac{\sin\angle FBD}{\sin\angle BDF} = \frac{FD}{BF}$$
$$\frac{\sin\angle EPF}{\sin\angle PEF} = \frac{EF}{PF}$$
$$\frac{\sin\angle FPD}{\sin\angle FDP} = \frac{FD}{FG}$$
$$\frac{\sin\angle DBM}{\sin\angle BDM} = \frac{DM}{BM}$$
$$\frac{\sin\angle BEM}{\sin\angle EBM} = \frac{BM}{EM}$$
$$\frac{\sin\angle DPM}{\sin\angle PDM} = \frac{DM}{PM}$$
$$\frac{\sin\angle MEP}{\sin\angle EPM} = \frac{PM}{EM}.$$

Multiplying and flipping the equations so that equal terms in the numerator and denominator cancel, we have $\frac{\sin\angle EBF}{\sin\angle FBD} = \frac{\sin\angle DBM}{\sin\angle EBM}$, and we conclude that $\angle EBF = \angle DBM = \angle DNM = 30^\circ$.

Then, we can see that F is the incenter of $\triangle BEG$ since $\triangle BEG$ is equilateral, F lies on the angle bisector EM, and $\angle EBF = 30^\circ$ as well. We have $FE = FG$, $FC = FP$ (they subtend the same angle measure in the circumcircle of $CEFP$), $\angle FCE = \angle FPG$, and $\angle FEC = \angle FGP$ since $\angle FEC = \angle CEP + \angle FEP = 150^\circ$ and $\angle FGP = 180^\circ - \angle FGE = 180^\circ - \angle FEG = 150^\circ$. This gives us $\triangle FEC \cong \triangle FGP$, so $GP = CE = 5$. Also, note that $\triangle BEG$ is equilateral so $EG = EB = 11$. Then, law of cosines on $\triangle CEP$ gives us $CP = \sqrt{5^2 + 16^2 - 2(5)(16)\cos(120^\circ)} = \boxed{19}$.