1. Kevin evaluates the sum of all positive divisors of 10000 that are multiples of 100 and writes the result on a blackboard. Underneath, he evaluates the sum of all positive divisors of 100 and writes down that result. Compute the ratio of the top number to the bottom number.

   Solution: Let D be the sum of the positive divisors of 100. A divisor of $10000 = 100^2$ that is a multiple of 100 can be written as $100d$ for some positive integer d satisfying $100d \mid 100^2 \iff d \mid 100$. That is, this divisor is 100 times a positive divisor of 100. Thus, the sum of the positive divisors of 10000 that are multiples of 100 can be written as $100D$. The answer is thus $100D/D = \boxed{100}$.

2. Let X be a 2024 digit perfect square. Let $a(X)$ be the 1012 digit number formed from reading the first 1012 digits of X, in order, and let $b(X)$ be the 1012 digit number formed from reading the last 1012 digits of X, in order. Given that X is the unique choice that maximizes $a(X) - b(X)$, find the sum of digits of X.

   Solution: Intuitively, we want the first half to have as many 9's as possible in terms of digits, and the second half as much 0's. This leads us to conjecture that letting $X = 999...9^2$ where there are 1012 occurrences of the digit 9 gives the optimal solution. Expanding, we get 99...9800....01 where there are 1011 occurrences of the digit 9 and 1011 occurrences of the digit 0. This gives a $a(X) - b(X)$ difference of 99...97. Computing adjacent perfect squares, we find X is optimal and unique. The answer is thus $9 \cdot 1011 + 8 + 1 = \boxed{9108}$.

3. Let F be a set of subsets of $\{1, 2, 3\}$. F is called *distinguishing* if each of 1, 2, and 3 are distinguishable from each other—that is, 1, 2, and 3 are each in a distinct set of subsets from each other. For example $F = \{\{1, 3\}, \{2, 3\}\}$ is *distinguishing* because 1 is in $\{1, 3\}$, 2 is in $\{2, 3\}$, and 3 is in $\{1, 3\}$ and $\{2, 3\}$. $F = \{\{1, 2\}, \{2\}\}$ is also *distinguishing*: 1 is in $\{1, 2\}$, 2 is in $\{1, 2\}$ and $\{2\}$, and 3 is in none of the subsets.

   On the other hand, $F = \{\{1\}, \{2, 3\}\}$ is not *distinguishing*. Both 2 and 3 are only in $\{2, 3\}$, so they cannot be distinguished from each other.

   How many *distinguishing* sets of subsets of $\{1, 2, 3\}$ are there?

   Solution: There are a total of 8 possible subsets of $\{1, 2, 3\}$, so there are $2^8 = 256$ possible collections of these subsets. We will solve this problem by complementary counting—determining how many sets are not distinguishable.

   There are two cases: either none of the values $\{1, 2, 3\}$ can be distinguished, or only one of the values $\{1, 2, 3\}$ can be distinguished.

   If none of the values can be distinguished, then the only possible subsets are $\{\}$ and $\{1, 2, 3\}$. Each of these subsets can be either included or excluded from F, so we have a total of $2^2 = 4$ such collections F.

   If only one of the values can be distinguished, say without loss of generality that this value is 1. Then, we are restricted to the subsets $\{\}$, $\{1, 2, 3\}$, $\{1\}$, $\{2, 3\}$ (since 2 and 3 cannot be distinguished from each other). Each of these subsets can be either included or excluded in F, so we have a total of $2^4 = 16$ such collections. However, 4 of these collections (the ones with only $\{\}$

and $\{1,2,3\}$ have already been counted in the first case. Thus, we have $16-4=12$ possible collections. Either 1, 2, and 3 can be distingushed, so we have a total of $12 \cdot 3 = 36$ total collections.

Our final answer is $256-4-36=\boxed{216}$.

4. Each vertex and edge of an equilateral triangle is randomly labelled with a distinct integer from 1 to 10, inclusive. Compute the probability that the number on each edge is the sum of those on its vertices.

   Solution: We count the number of working configurations first. Assume the triangle is in upright position, and let x, y, z be the labels of the vertex on the top, bottom-left, and bottom-right, respectively. Without a loss of generality, assume $x > y > z$ (we will multiply by 6 later). Now, note that all edge labels are fixed, and distinct, since $x+y > x+z > y+z$. Thus, the only "repeats" can occur between a vertex and an edge. But an edge clearly cannot repeat value with any of its vertices, and so may only repeat value with the opposite vertex. This occurs when some two of x, y, z sum to the other, which since $x > y > z$ can only occur when $x = y + z$. There is one more condition we need to worry about: because all labels are from $\{1, .., 10\}$, we have $10 \geq x + y > x + z > y + z$. Now, since $y \neq 1, 10$ (being the middle value), we can fix y from 2 to 9. Then there are $y - 1$ possible values of z, and $10 - 2y$ possible values of x (from $y < x \leq 10 - y$.) This gives $(y-1)(10-2y)$ total pairs. Summing over all y, we get a total of $1 \cdot 6 + 2 \cdot 4 + 3 \cdot 2 + 0 + \cdots + 0 = 20$ total labelings. However, we need to subtract those with $x = y + z$. To compute this, we can do casework on x. As x varies from 3 to 10, the number of $y + z = x$ with $y > z$ is $1, 1, 2, 1$, from a quick manual inspection, giving a total of $20 - 1 - 1 - 2 - 1 = 15$ valid labelings. Thus there are $15 \times 6 = 90$ total labelings, without the assumption that $x > y > z$. There are $10 \times 9 \times 8 \times 7 \times 6 \times 5$ total orderings, giving an answer of $\frac{90}{10\times9\times8\times7\times6\times5} = \boxed{\frac{1}{1680}}$.

5. We define the *spillage* of a number as $s(x) = \left\lfloor \frac{x}{100} \right\rfloor$, that is, the largest integer that is at most $\frac{x}{100}$. The *spillage* of a list of numbers $[a_1, a_2, ..., a_n]$ is the sum of left to right *spillages*: $s(a_1) + s(a_1a_2) + s(a_1a_2a_3) + ... + s(a_1a_2 \cdots a_n)$. Let M be the minimum possible *spillage* of $[1, 2, ..., 10]$ over all the permutations of this list. How many of these permutations achieve M?

   Solution: Note that the minimum possible *spillage* can be achieved by minimizing the sum of the terms $a_1, a_1a_2, ...,$ and $a_1a_2\cdots a_n$. One ordering that minimizes the sum simply places the largest values in the list in the least number of products. For our list, this permutation is $\pi = [1, 2, ..., 10]$. The last *spillage* value in the sum is fixed as $s(1 \cdot 2 \cdots 10)$ for any permutation.

   We now consider which numbers can be moved in the ordering while still achieving the minimum *spillage* sum. If $a_{10} \neq 10$, then $s(a_1a_2\cdots a_9) > s(1 \cdot 2 \cdots 9)$ and the minimum that can be achieved is to have the rest of the *spillage* values remain the same as in permutation π. Thus, the minimum *spillage* sum cannot be achieved if we do not set $a_{10} = 10$. We can use the same reasoning to conclude that we must set $a_9 = 9, a_8 = 8$, and $a_7 = 7$. For a_6, we see that in the permutation π, the first 6 terms in the *spillage* sum are $0, 0, 0, 0, 1, 7$. Since we have fixed a_{10} through a_7, we know that $a_1a_2\cdots a_6 = 720$, which gives $s(a_1a_2\cdots a_6) = 7$. We want to choose a_6 so that $s(a_1a_2\cdots a_5) = 1$, which requires that $a_6 \geq 4$. Then, we want to choose a_5 so that $s(a_1a_2a_3a_4) = 0$, which requires that $a_5a_6 \geq 8$. The remaining numbers can be ordered in any way. For each of the values $4, 5$, and 6 for a_6, there are 4 possible values for a_5 so that $a_5a_6 \geq 8$. Then, there are $4!$ ways to order the remaining numbers. Thus, the number of permutations that achieve the minimum *spillage* sum is $3 \cdot 4 \cdot 4! = \boxed{288}$.

6. Alice is playing with magnets on her fridge. She has 7 magnets, with the numbers $1, 2, 3, 4, 5, 6, 7$, in that order in a row, and she also has two magnets with a "$+$" sign, two magnets with a "$-$" sign, and two magnets with a "$\times$" sign. She randomly puts these six operation magnets between her 7 number magnets, with one operation between every two consecutive numbers, and evaluates the resulting expression (following the order of operations). What is the expected value of her result?

   Solution: Suppose that Alice instead has a list of 5 numbers a_1, a_2, a_3, a_4, a_5, two plus, and two minus magnets. Then, the expected result of placing these magnets is just a_1: by linearity of expectation, there is a $\frac{1}{2}$ probability of each of a_2, a_3, a_4, a_5 being added or subtracted from the result.

   Hence, there are only three cases for the answer: $1, 2, 6$ depending on where the multiplication symbols are placed. There are a total of $\frac{6!}{2!2!2!} = 90$ ways of placing the magnets. Of these, 6 of them have the term $1 \times 2 \times 3$ and $4 \cdot 6 = 24$ of them have the term 1×2. Thus, the answer is

$$6 \cdot \frac{6}{90} + 2 \cdot \frac{24}{90} + 1 \cdot \frac{90 - 6 - 24}{90} = \boxed{\frac{8}{5}}.$$

7. Let $1 \le A \le 119$ and $1 \le B \le 139$ be two integers such that $\frac{A}{60}$ and $\frac{B}{70}$ are fractions in simplest form, yet, when adding $\frac{A}{60}$ and $\frac{B}{70}$ by rewriting both fractions with their lowest common denominator and adding the resulting numerators, the new fraction can be simplified. Find how many ordered pairs (A, B) are possible.

   Solution: Note that $\gcd(A, 60) = \gcd(B, 70) = 1$ and that $\frac{7A+6B}{420}$ is the summand. Since this can be simplified, we know $7A + 6B$ and 420 must share some common divisor larger than 1. Since $\gcd(A, 60) = \gcd(B, 70)$, $7A + 6B$ is relatively prime to 2, 3, and 7. It follows that $7A + 6B$ must be a multiple of 5, the only remaining prime factor of 420. Thus $2A + B \equiv 0 \pmod 5$, implying $B \equiv -2A \pmod 5$. Now, $\varphi(60) = 16$, and $\varphi(70) = 24$. This means there are $2 \cdot 16 \cdot 24 \cdot 2$ ways to choose A, B, without regards to restrictions on $\pmod 5$. However, with the restriction, once A is chosen B is fixed $\pmod 5$. Since the possible values of B are symmetric for $1, 2, 3, 4 \pmod 5$, we can divide by 4 and obtain an answer of $16 \cdot 24 = \boxed{384}$.

8. Call a polynomial P *cool* if it has degree less than 257, each of its coefficients are nonnegative integers less than 257, and

$$\sum_{k=0}^{256} P(k^j)$$

   is divisible by 257 for all positive integers j. How many *cool* polynomials are there? (Assume that the polynomial $P(x) = 0$ has degree less than 257.)

   Solution: Let $p = 257$. Note that 257 is prime (which can be verified by checking that no prime up to 13 is a divisor). Suppose

$$P(x) = a_{p-1}x^{p-1} + \cdots + a_1 x + a_0$$

is *cool*. Then we have

$$\sum_{i=0}^{p-1} a_i \left(\sum_{k=0}^{p-1} k^{ji} \right)$$

is a multiple of p for all positive integers j. Also, by Fermat's Little Theorem we only have to consider j at most $p-1$.

Note that $\displaystyle\sum_{k=0}^{p-1} k^{ji}$ is a multiple of p if and only if ji is not a multiple of $p-1$, and is $-1 \pmod p$ if $(p-1) \mid ji$. To see this, note that when $(p-1) \mid ji$, we have $\sum_{k=0}^{p-1} k^{ji} \equiv \sum_{k=1}^{p-1} 1 \equiv -1 \pmod p$. When $(p-1) \nmid ji$, let g be a generator of the multiplicative group $\mathbb{Z}_p^* = \{1, \dots, p-1\}$, so we have $g^{ji} \not\equiv 1 \pmod p$. We see that $g, 2g, \dots, (p-1)g$ are all distinct elements of $\mathbb{Z}_p^*$, so

$$1^{ji} + \dots + (p-1)^{ji} \equiv g^{ji} + \dots((p-1)g)^{ji} \equiv g^{ji}(1^{ji} + \dots + (p-1)^{ji}).$$

Since $g^{ji} \not\equiv 1 \pmod p$, this implies that $1^{ji} + \dots + (p-1)^{ji} \equiv 0 \pmod p$.

Thus, letting $d = \frac{p-1}{\gcd(p-1,j)}$, we have

$$p \mid a_d + a_{2d} + \dots + a_{p-1}$$

for all $d \mid p-1$.

Now we use $p = 257$. We have $a_{256} = 0$ by taking $d = 256$. Now take $d = 128$, to get 257^0 possibilities for a_{128}. Then, $d = 64$ gives us 257^1 possibilities for a_{64} and a_{192}. Continuing, taking $d = 2^k$ will give $257^{2^{7-k}-1}$ possibilities for $a_d, a_{3d}, \dots$, since we can freely choose the first $2^{7-k} - 1$ elements in the sequence, leaving one possibility for the last element. Additionally, we have no restrictions on a_0, so we multiply by 257. Thus, the number of solutions is

$$257 \cdot 257^0 \cdot 257^1 \cdot 257^3 \cdots \cdot 257^{127} = \boxed{257^{248}}.$$

In general, the answer is $p^{p-d(p-1)}$ where $d(x)$ is the number of divisors of x.

9. Let $f : \mathbb{N} \to \mathbb{R}$ be a function which satisfies $\frac{1}{64}n^2 = \sum_{d|n} f(d)f(\frac{n}{d})$. What is the least integer n for which $f(n)$ is an integer?

Solution: First of all, define $g(n) = 8\frac{f(n)}{n^2}$. Then, $g(1) = 1$, and g is multiplicative over coprimes, i.e. if m and n are coprime then $g(mn) = g(m)g(n)$. To see this, we can induct on mn (the base case $m = n = 1$ is clear). Then,

$$\begin{aligned}
1 &= \sum_{d|mn} g(d)g\left(\frac{mn}{d}\right) \\
&= 2g(mn) + \sum_{1<d<mn} g(d)g\left(\frac{mn}{d}\right) \\
&= 2g(mn) + \sum_{\substack{1<dd'<mn \\ d|m, d'|n}} g(d)g(d')g\left(\frac{m}{d}\right)g\left(\frac{n}{d'}\right) \\
&= 2g(mn) - 2g(m)g(n) + \sum_{d|m, d'|n} g(d)g(d')g\left(\frac{m}{d}\right)g\left(\frac{n}{d'}\right) \\
&= 2g(mn) - 2g(m)g(n) + \left(\sum_{d|m} g(d)g\left(\frac{m}{d}\right)\right)\left(\sum_{d'|n} g(d')g\left(\frac{n}{d'}\right)\right) \\
&= 2g(mn) - 2g(m)g(n) + 1,
\end{aligned}$$

so $g(mn) = g(m)g(n)$, as desired.

Here, we have used the fact that since m and n are coprime, then d and d' are coprime, as well as $\frac{m}{d}$ and $\frac{n}{d'}$.

Let us consider the sequence of numbers c_k defined by $c_k = g(p^k)$ (so, $c_0 = 1, c_1 = \frac{1}{2}, \cdots$). We see that,

$$\left(\sum_{k\geq 0} c_k x^k\right)^2 = \sum_{n\geq 0} x^n \sum_{a+b=n} c_a c_b = \sum_{n\geq 0} x^n \sum_{a+b=n} g(p^a)g(p^b) = \sum_{n\geq 0} x^n = \frac{1}{1-x}.$$

This gives us $\sum_{k\geq 0} c_k x^k = \pm(1-x)^{-1/2}$. We will proceed with $\sum_{k\geq 0} c_k x^k = (1-x)^{-1/2}$, as the cases are symmetrical and will result in the same answer. We have $c_k = (-1)^k \binom{-\frac{1}{2}}{k}$.

After expanding the expression, we get $c_k = \frac{\binom{2k}{k}}{2^{2k}}$. In particular, we get that $f(p^k) = \frac{p^{2k}}{8} \cdot \frac{\binom{2k}{k}}{2^{2k}}$. Since we're only looking for the least integer n for which $f(n)$ is an integer, we may assume that n is a power of 2. Write $n = 2^k$, then $f(n) = \frac{\binom{2k}{k}}{8}$. Thus this whole thing comes down to finding minimal k such that $\frac{\binom{2k}{k}}{8}$ is an integer. It turns out, either via brute force or via degree-counting in factorials, that $k = 7$ is the desired answer. Therefore, $n = \boxed{128}$.

10. There are $2n$ students taking an exam, and at the beginning they all put their phones into a pile. When leaving, each person takes an arbitrary phone from the pile. Unfortunately, it might be the case that some students did not get back their own phone!

To get back the correct phones, the students come up with the following strategy. They repeat the following *round* as many times as needed:

1. Some of the students pair up. Each student can be in at most one pair.

2. The pairs swap phones according to some swap order (i.e. an ordering of the pairs).

For a given assignment π of the $2n$ students to the phones they originally picked up, let $r(\pi)$ be the minimum number of rounds required for the students to each receive back their own phone, assuming the students make swaps optimally. Let $s(\pi)$ be the number of ways to swap phones (determined by pairings and swap orders over all rounds) achieving $r(\pi)$ rounds. Let $M(n)$ be the maximum value of $r(\pi)$ over all assignments π, and let $f(n)$ be the sum of $s(\pi)$ over all π with $r(\pi) < M(n)$.

Then, there exists a unique ordered pair (a, b) with $a > 0$ and $b > 0$ such that $\lim_{n\to\infty} \frac{f(n)\cdot a^n}{(2n)!} = b$. Compute (a, b).

Note: It may be helpful to know that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Solution: We claim that for any assignmenr π, the value of $r(\pi)$ is at most 2. Indeed, consider the cycle decomposition $\sigma_1 \cdot \sigma_2 \cdots \sigma_k$ of the permutation π corresponding to the student-phone pairs. First, the answer is 0 whenever this is the identity permutation. Next, each transposition (cycle of length 2) can be dealt with in one round in parallel. So, it suffices to just consider the cycles of length at least 3.

We will deal with each such cycle in parallel in two rounds, thus giving the desired result. Indeed, consider WLOG a cycle with $\pi(1) = 2, \pi(2) = 3, ..., \pi(r-1) = r, \pi(r) = 1$ (student 1 has student 2's phone, student 2 has student 3's phone, etc.)

Then, consider swapping the phones that student 1 and $k-1$ are holding, the ones 2 and $k-2$ are holding, and so on through $\left\lfloor \frac{k-1}{2} \right\rfloor$ and $\left\lceil \frac{k}{2} \right\rceil$ (student k and possibly one student in the middle do not participate in any swaps). Then, this pairs up $\pi(1) = k, \pi(k) = 1$, and $\pi(2) = k-1, \pi(k-1) = 2$, and so on. From here, the problem is reduced to one on transpositions so the answer must be at most 2.

Now, the answer is at most 1 if and only if the maximum cycle size is at most 2. The sum of $s(\pi)$ over these is $f(n) = \sum_{k=0}^{n} \binom{2n}{2k} \cdot \frac{(2k)!}{2^k}$ (choose the $2k$ students in transpositions and then pair them up). We can rewrite this as

$$\begin{aligned}
f(n) &= \sum_{k=0}^{n} \binom{2n}{2k} \cdot \frac{(2k)!}{2^k} \\
&= (2n)! \sum_{k=0}^{n} \frac{1}{2^k \cdot (2n-2k)!} \\
&= (2n)! \sum_{k=0}^{n} \frac{2^{-k}}{(2n-2k)!} \\
&= \frac{(2n)!}{2^n} \sum_{k=0}^{n} \frac{(\sqrt{2})^{2n-2k}}{(2n-2k)!} \\
&= \frac{(2n)!}{2^n} \sum_{k=0}^{n} \frac{(\sqrt{2})^{2k}}{(2k)!}
\end{aligned}$$

Let us recall that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ and thus $\frac{e^x + e^{-x}}{2} = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}$. Thus, as $n \to \infty$, we can rewrite the above as

$$f(n) \to \frac{(2n)!}{2^n} \cdot \left(\frac{e^{\sqrt{2}} + e^{-\sqrt{2}}}{2} \right)$$

or equivalently

$$\frac{f(n) \cdot 2^n}{(2n)!} \to \left(\frac{e^{\sqrt{2}} + e^{-\sqrt{2}}}{2}\right).$$

Our answer is $\boxed{\left(2, \frac{e^{\sqrt{2}}+e^{-\sqrt{2}}}{2}\right)}$.