1. Compute $f'(0)$ if

$$f(x) = \frac{6x - x^3 + 5x^4 + e^x}{5 + 3x^2 + 2x^3 + \cos(x)}.$$

Solution: Note that we only need to evaluate $f'(0)$ instead of $f'(x)$.

Let $f(x) = \frac{p(x)}{q(x)}$. By the quotient rule, $f'(x) = \frac{p'(x)q(x) - p(x)q'(x)}{q^2(x)}$. Notice that $q'(0) = 0$. Then, $f'(0) = \frac{p'(0)}{q(0)} = \boxed{\frac{7}{6}}$.

2. A curve contained in the first quadrant of the xy-plane originates from $(1, 0)$ and has the following property: at each point (x, y) on the curve, the segment of the tangent line connecting (x, y) to the intersection of the tangent with the y-axis has length 1.

What is the area of the region bounded by the curve, the x-axis, and the y-axis?

Solution: First, we compute the equation of the curve. Suppose the curve is the image of $f(x)$. We see that the tangent line at the point $(x, f(x))$ intersects the y-axis at the point $(0, f(x) - xf'(x))$. The distance is then $x\sqrt{(f'(x))^2 + 1}$. We therefore have the following equation:

$$f'(x)^2 + 1 = \frac{1}{x^2}.$$

This gives us $f'(x) = -\frac{\sqrt{1-x^2}}{x}$ (note that the negative sign is necessary in order for the curve to be in the first quadrant). Therefore, for each $0 < t \leq 1$, we have $f(t) = \int_t^1 \frac{\sqrt{1-x^2}}{x}\,dx$. We proceed to show how this integral can be evaluated.

Standard solution: There are many ways to compute this integral. One way is to make the u-substitution $u = \sqrt{1 - x^2}$ and then use partial fraction decomposition (but do be careful that $0 < x < 1$). Regardless, the result is $f(x) = -\sqrt{1 - x^2} + \frac{1}{2}\log\left(\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right)$. Note that this is a curve that originates from $(1, 0)$ and travels left and up, approaching infinity in height as x approaches 0.

The next step is to integrate $f(x)$ from 0 to 1. Note that $\int_0^1 \sqrt{1 - x^2}\,dx = \frac{\pi}{4}$. The hard part is what to do with $\frac{1}{2}\log\left(\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right)$. We will first rewrite this integrand, with some log rules, as

$$\frac{1}{2}\log\left(\frac{\left(1 + \sqrt{1 - x^2}\right)^2}{x^2}\right) = \log(1 + \sqrt{1 - x^2}) - \log(x).$$

The antiderivative of $\log(x)$ is $x\log(x) - x$; evaluating from 0 to 1 (with the evaluation at 0 taken to mean a limiting action), we get -1. Finally, there is $\log\left(1 + \sqrt{1 - x^2}\right)$. Using integration by parts, we get

$$\int \log\left(1 + \sqrt{1 - x^2}\right)dx = x\log\left(1 + \sqrt{1 - x^2}\right) + \int \frac{x}{1 + \sqrt{1 - x^2}} \cdot \frac{x}{\sqrt{1 - x^2}}\,dx.$$

But note that $\frac{x}{1+\sqrt{1-x^2}} \cdot \frac{x}{\sqrt{1-x^2}} = \frac{x^2}{\left(1+\sqrt{1-x^2}\right)\left(1-\sqrt{1-x^2}\right)} \cdot \frac{1-\sqrt{1-x^2}}{\sqrt{1-x^2}} = -1 + \frac{1}{\sqrt{1-x^2}}$, so the entire antiderivative ends up being

$$x\log\left(1 + \sqrt{1 - x^2}\right) - x + \arcsin(x) + C.$$

Evaluating at 0 and 1 gets $\frac{1}{2}(\pi-2)$. Putting this all together, the area under the curve is

$$-\frac{\pi}{4}-1+\frac{1}{2}\pi-1=\boxed{\frac{\pi}{4}}.$$

However, to those who do know double integrals, there is a much quicker solution.

Alternate solution: We seek the value of $\int_0^1\int_t^1 \frac{\sqrt{1-x^2}}{x}\,dx\,dt$. Non-negativity of the integrand justifies Fubini. Switching the order of integration, the integral becomes $\int_0^1\int_0^x \frac{\sqrt{1-x^2}}{x}\,dt\,dx = \int_0^1 \sqrt{1-x^2}\,dx = \boxed{\frac{\pi}{4}}$.

3. Compute

$$\sum_{n=0}^{\infty}\left(\int_0^{\pi}\sin^{2n}(x)\,dx\int_0^{\pi}\sin^{2n+1}(x)\,dx\right)^2.$$

Solution: Let $f_k=\int_0^{\pi}\sin^k(x)dx$. We quickly find that $f_0=\pi$ and $f_1=2$. For $k\geq 2$, we can find a recurrence relation:

$$\begin{aligned}
f_k &= \int_0^{\pi}\sin^k(x)dx\\
&= \int_0^{\pi}\sin^2(x)\sin^{k-2}(x)\,dx\\
&= \int_0^{\pi}(1-\cos^2(x))\sin^{k-2}(x)\,dx\\
&= \int_0^{\pi}\sin^{k-2}(x)\,dx-\int_0^{\pi}\cos^2(x)\sin^{k-2}(x)\,dx\\
&= f_{k-2}-\int_0^{\pi}\cos(x)(\cos(x)\sin^{k-2}(x))\,dx\\
&= f_{k-2}-\int_0^{\pi}udv,
\end{aligned}$$

where $u=\cos(x)$ and $v=\frac{1}{k-1}\sin^{k-1}(x)\implies dv=\cos(x)\sin^{k-2}(x)dx$. By applying integration by parts, we have

$$\begin{aligned}
f_k &= f_{k-2}-\left(uv\,\Big|_0^{\pi}-\int_0^{\pi}duv\right)\\
&= f_{k-2}+\int_0^{\pi}-\sin(x)\frac{1}{k-1}\sin^{k-1}(x)\,dx-\left(\frac{\cos(x)\sin^{k-1}(x)}{(k-1)}\,\Big|_0^{\pi}\right).
\end{aligned}$$

The last term cancels to zero because $\sin(x)=0$ at $x=0,\pi$. So, we end up with

$$f_k = f_{k-2} - \frac{1}{k-1}\int_0^\pi \sin^k(x)\,dx$$
$$= f_{k-2} - \frac{1}{k-1} f_k.$$

Rearranging, we get $f_k = \frac{k-1}{k} f_{k-2}$ Thus, for even $2k$, $f_{2k} = \pi \cdot \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2k-1}{2k}$ and for odd $2k+1$, $f_{2k+1} = 2 \cdot \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2k}{2k+1}$.

Thus, $f_{2k} \cdot f_{2k+1} = 2\pi \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{2k}{2k+1} = \frac{2\pi}{2k+1}$. Our final sum is

$$\sum_{n=0}^{\infty} (f_{2k} \cdot f_{2k+1})^2 = \sum_{n=0}^{\infty} \left(\frac{2\pi}{2k+1}\right)^2$$
$$= 4\pi^2 \cdot \left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots\right).$$

It is well known that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots = \frac{\pi^2}{6}$. So, $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) - \left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \frac{1}{8^2} + \ldots\right) = \frac{\pi^2}{6} - \frac{1}{4} \cdot \frac{\pi^2}{6} = \frac{\pi^2}{8}$. Thus our final sum is $4\pi^2 \cdot \frac{\pi^2}{8} = \boxed{\frac{\pi^4}{2}}$.