1. Compute the number of solutions $x \in [0, 2\pi]$ to

$$\cos(x^2) + \sin(x^2) = 0.$$

Solution: First observe that $\cos(x^2) + \sin(x^2) = 0 \implies \tan(x^2) = -1 \implies x^2 = -\frac{\pi}{4} + n\pi$. Since $x^2 \le 4\pi^2$, then $4\pi^2 \ge -\frac{\pi}{4} + n\pi \implies n \le 4\pi + \frac{1}{4}$. With a minimal bit of arithmetic, we see that $n \le 12$. But $n \ge 1$, since if $n = 0$ then $x^2 < 0$. Therefore, there are $\boxed{12}$ solutions.

2. Suppose that for some angle $0 \le \theta \le \pi/4$, the roots of $x^2 + ax + \frac{3}{10}$ are $\sin\theta$ and $\cos\theta$. If the roots of $p(x) = x^2 + cx + d$ are $\sin(2\theta)$ and $\cos(2\theta)$, what is the value of $p(1)$?

Solution: From Vieta's formulas, we have $\sin\theta\cos\theta = \frac{3}{10}$. Then, $\sin(2\theta) = 2\sin\theta\cos\theta = \frac{3}{5}$, which gives us $\cos(2\theta) = \pm\frac{4}{5}$. Since $0 \le \theta \le \pi/4$, we know $0 \le 2\theta \le \pi/2$, which means $\cos(2\theta) = \frac{4}{5}$. Using Vieta's again we have

$$\begin{aligned} p(1) &= 1 - (\sin(2\theta) + \cos(2\theta)) + \sin(2\theta)\cos(2\theta) \\ &= 1 - \left(\frac{3}{5} + \frac{4}{5}\right) + \frac{3}{5} \cdot \frac{4}{5} \\ &= \boxed{\frac{2}{25}}. \end{aligned}$$

3. Let $q(x) = x^3 - 9x^2 + 18x + 27$. Compute

$$q(-10) + q(-8) + q(-6) + \ldots + q(16).$$

Solution: Note that $x^3 - 9x^2 + 18x = x(x-3)(x-6)$, so $q(x) + q(6-x) = x(x-3)(x-6) + 27 + (6-x)(3-x)(-x) + 27 = 54$. We can pair the terms $q(-10)$ and $q(16)$, $q(-8)$ and $q(14)$, and so on. The sum contains 7 pairings of $q(x)$ and $q(6-x)$, so the answer is $7 \cdot 54 = \boxed{378}$.

4. Compute

$$\frac{5}{1\cdot2\cdot3\cdot4} + \frac{7}{2\cdot3\cdot4\cdot5} + \ldots + \frac{99}{48\cdot49\cdot50\cdot51}.$$

Solution: Note that $\frac{1}{x^2-1} - \frac{1}{(x+1)^2-1} = \frac{2x+1}{(x-1)x(x+1)(x+2)}$, so we can write the sum as

$$\frac{1}{2^2-1} - \frac{1}{3^2-1} + \frac{1}{3^2-1} - \frac{1}{4^2-1} + \ldots + \frac{1}{49^2-1} - \frac{1}{50^2-1} = \frac{1}{3} - \frac{1}{2499} = \boxed{\frac{832}{2499}}.$$

5. Ashley writes the concatenation of $\lfloor 2.5^1 \rfloor, \lfloor 2.5^2 \rfloor, \ldots, \lfloor 2.5^{1000} \rfloor$ on the board. Her number is 199667 digits long. Now, Bob writes the concatenation of $4^1, 4^2, \ldots, 4^{1000}$ on the board. Compute the number of digits in Bob's number.

Solution: The expressions in the following assume a log base of 10. We know that the number of total digits in $\lfloor 2.5^x \rfloor$ and 4^x is $\lceil \log 2.5^x \rceil + \lceil \log 4^x \rceil$. Since we know that $\log 2.5^x$ and $\log 4^x$ are not integers, and $\log 2.5^x + \log 4^x = x \log 10 = x$, $\lceil \log 2.5^x \rceil + \lceil \log 4^x \rceil = x + 1$. Then, the total number of digits in the Ashley's and Bob's numbers Is $2 + 3 + \ldots + 1001 = \frac{(1000)(1003)}{2} = 501500$. Then, the number of digits in Bob's number is $501500 - 199667 = \boxed{301833}$.

6. Let $f(x, y) = xy$ and $g(x, y) = x^2 - y^2$. If a counterclockwise rotation of θ radians about the origin sends $g(x, y) = a$ to $f(x, y) = b$, compute the value of $\frac{a}{b\tan\theta}$.

   Solution: It's easier to think of the problem visually. For instance, we can already make the correct guess for θ just by noticing that the axes of symmetry for $xy = m$, for any nonzero integer m, are $y = \pm x$. And that the axes of symmetry for for $x^2 - y^2 = n$, for any nonzero integer n, are the coordinate axes. The axes of symmetry for $xy = m$ are 45-degree rotations of the axes of symmetry for $x^2 - y^2 = n$, regardless of the values of m and n. Thus, $\theta = \frac{\pi}{4}$. There is no need to worry about displacement, since both curves are centered at the origin. The problem in other words is finding a relation between a and b so that each hump of $xy = b$ and $x^2 - y^2 = a$ is the same distance from the origin. The distance between either vertex of the hyperbola, which are $(\pm\sqrt{a}, 0)$, and the origin is $\sqrt{a}$. The distance between either hump of the rational function, which are $(\pm\sqrt{b}, \pm\sqrt{b})$ is $\sqrt{2b}$. Thus, $\sqrt{a} = \sqrt{2b}$ and $a = 2b$. Thus, plugging in this relation and the value of θ, we get $\frac{2b}{b\tan\frac{\pi}{4}}$, which is equal to $\boxed{2}$.

7. Find the number of lines of symmetry that pass through the origin for

   $$|xy(x + y)(x - y)| = 1.$$

   Solution: Using polar coordinates, let $(x, y) = (r\cos(t), r\sin(t))$. It follows that the equation in the question reduces through this to

   $$\frac{1}{4}\sin(4t)r^4 = \pm 1$$

   If the line with angle $t = \theta$ is a line of symmetry, then it follows that if $(r, t) = (r, \theta + \epsilon)$ satisfies the above equation, then so must $(r, t) = (r, \theta - \epsilon)$. Therefore,

   $$\left|\frac{1}{4}\sin(4(\theta + \epsilon))r^4\right| = \left|\frac{1}{4}\sin(4(\theta - \epsilon))r^4\right|$$

   for any ϵ. This is only possible if $\sin(4t) = 0$ or $\sin(4t) = \pm 1$. We now find all values of $t \in [0, 2\pi)$, since any other values will result in the same line of symmetry as one already found. The possible values of t are $0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \frac{3\pi}{4}$, and $\frac{7\pi}{8}$. Therefore, there are $\boxed{8}$ lines of symmetry.

8. Let a_1, a_2, a_3, and a_4 be non-negative real numbers such that $a_1^2 + a_2^2 + a_3^2 + a_4^2 = 1$. Compute the minimum possible value of

   $$6a_1^3 + 8a_2^3 + 12a_3^3 + 24a_4^3.$$

   Solution: We have $6a_1^3 + 8a_2^3 + 12a_3^3 + 24a_4^3 = 24\left(\frac{a_1^3}{4} + \frac{a_2^3}{3} + \frac{a_3^3}{2} + \frac{a_4^3}{1}\right)$. We then have that

   $$\frac{a_1^3}{4} + \frac{a_2^3}{3} + \frac{a_3^3}{2} + \frac{a_4^3}{1} \geq \frac{(a_1^2 + a_2^2 + a_3^2 + a_4^2)^{3/2}}{(4^2 + 3^2 + 2^2 + 1^2)^{1/2}} = \frac{1}{\sqrt{30}},$$

   by Radon's Inequality (which can be derived from Hölder's Inequality). Our answer is $\frac{24}{\sqrt{30}} = \boxed{\frac{4\sqrt{30}}{5}}$. This is achieved when $a_1 = \frac{4}{\sqrt{30}}, a_2 = \frac{3}{\sqrt{30}}, a_3 = \frac{2}{\sqrt{30}}$, and $a_4 = \frac{1}{\sqrt{30}}$.

9. Compute

$$\sum_{a=0}^{100}\sum_{b=0}^{100}\frac{1}{1+\cos\left(\frac{2\pi(a-b)}{101}\right)}.$$

Solution: Let $\omega = e^{i\theta}$. We may expand the denominator to find that it is $\frac{1}{2}((\cos(a\theta)+\cos(b\theta))^2 + (\sin(a\theta)+\sin(b\theta))^2)$, which is equal to $\frac{1}{2}(\omega^a+\omega^b)(\omega^{-a}+\omega^{-b})$ or equivalently $\frac{1}{2}\cdot\frac{(\omega^a+\omega^b)^2}{\omega^{a+b}}$. Thus, the sum is equal to

$$\sum_{a=0}^{100}\sum_{b=0}^{100}\frac{1}{1+\cos((a-b)\theta)} = 2\sum_{a=0}^{100}\omega^a\sum_{b=0}^{100}\frac{\omega^b}{(-\omega^a-\omega^b)^2}.$$

Consider the polynomial $p(x) = x^{101}-1$ with roots being the roots of unity $\omega_0, \omega_1, \ldots, \omega_{100}$. Then, for a given x we wish to compute $q(x) = \sum_{i=0}^{100}\frac{\omega_i}{(x-\omega_i)^2}$: then, plugging in $x = -\omega^a$ and summing over all a would give the result.

To compute $q(x)$, let us rewrite it as

$$q(x) = \sum_{i=0}^{100}\frac{1}{\omega_i - x} + x\sum_{i=0}^{100}\frac{1}{(\omega_i-x)^2}.$$

Now, consider the polynomial $r(z) = (z+x)^{101}-1$. Since the roots are $\omega_i - x$, it follows that we can also express the squares of the roots via

$$s(z^2) \triangleq \prod_{i=0}^{100}(z^2-(\omega_i-x)^2) = -\prod_{i=0}^{100}(z-(\omega_i-x))(-z-(\omega_i-x)) = -r(z)r(-z)$$

Now, let $[k]r(z)$ denote the coefficient of z^k in $r(z)$. Then, we immediately obtain that

$$\begin{aligned}
q(x) &= -\frac{[1]r(z)}{[0]r(z)} + x\cdot\frac{[1]s(z)}{[0]s(z)} \\
&= -\frac{[1]r(z)}{[0]r(z)} + x\cdot\frac{[2]s(z^2)}{[0]s(z)} \\
&= \frac{-101x^{100}}{x^{101}-1} - x\cdot\frac{-101^2x^{200}+101\cdot100\cdot x^{99}(x^{101}-1)}{(x^{101}-1)^2} \\
&= 101^2\cdot\left(\frac{x^{201}}{(x^{101}-1)^2} - \frac{x^{100}}{x^{101}-1}\right).
\end{aligned}$$

Since $x = -\omega^a$ satisfies $p(x) = -2$, $x^{100} = \omega^{-a}$, and $x^{201} = -\omega^{-a}$, we can simplify the above to

$$q(x) = \frac{101^2\omega^{-a}}{4}.$$

Thus, the answer is

$$2\cdot\sum_{a=0}^{100}\omega^a\cdot\frac{101^2\omega^{-a}}{4} = \frac{101^3}{2} = \boxed{\frac{1030301}{2}}.$$

Addendum: computing $q(x)$ with calculus. Consider $\frac{p'(x)}{p(x)} = \sum_{i=0}^{100} \frac{1}{x-\omega_i}$. We see then that $\sum_{i=0}^{100} \frac{x}{\omega_i - x} = -\frac{x \cdot p'(x)}{p(x)}$. Thus, since $\left(\frac{x}{\omega_i - x}\right)' = \frac{\omega_i}{(\omega_i - x)^2}$, it follows that

$$\sum_{i=0}^{100} \frac{\omega_i}{(x-\omega_i)^2} = \left(-\frac{x \cdot p'(x)}{p(x)}\right)' = -\frac{p'(x)}{p(x)} + \frac{xp'(x)^2}{p(x)^2} - \frac{xp''(x)}{p(x)}.$$

10. Call a polynomial $x^8 + b_7x^7 + \cdots + b_1x^1 + 1$ *binary* if each b_i is either 0 or 1. Compute the number of binary polynomials that have at least one real root.

Solution: Note that this polynomial cannot have any positive real roots, as all the terms are positive. Therefore, let us plug in $-x$ to get the polynomial $p(x) = x^8 - b_7x^7 + b_6x^6 - \cdots + b_2x^2 - b_1x_1 + 1$. The claim is that there exists a real root of this polynomial if and only if the number of odd values of i such that $b_i = 1$ is at least two more than the number of even values of i such that $b_i = 1$ (equivalently, the *total* number of even entries is at most the total number of odd entries, as x^8 and 1 are already fixed). Therefore, the answer is

$$\binom{4}{4} \cdot \left(\binom{3}{2} + \binom{3}{1} + \binom{3}{0}\right) + \binom{4}{3} \cdot \left(\binom{3}{1} + \binom{3}{0}\right) + \binom{4}{2} \cdot \binom{3}{0} = 7 + 16 + 6 = \boxed{29}.$$

Indeed, note that if there are at least as many odd as even entries, then $p(1) \leq 0$ and $p(0) = 1$. This implies there must exist a root between 0 and 1.

Else, suppose there are less odd than even entries. The proof of this section is long and tedious, and requires some careful approximation of values of the polynomial at certain points. In fact, the proof breaks down when the total degree is 10 or higher (consider $x^{10} - x^9 - x^7 + x^2 + 1$, which does have a real root).

Then, we first claim that there cannot exist a real root in the range $[0, 1]$. Note to begin that $p(0) = 1$ and $p(1) \geq 1$. In this range, the higher the power of x, the less it contributes. To check if there is a root, we determine whether the minimum value of the polynomial is at most 0. Hence, for any combination of a even entries and b odd entries, we can reduce to having the a even entries as high as possible (starting from 6 going down) and the b odd entries as low as possible (starting from 1 going up). Furthermore, we may assume that $a = b - 1$ (having more even entries will only make $p(x)$ take on larger values) and that the even and odd entries need not cross (as in, $-x^5 + x^4$ can be reduced by removing these two terms, as $x^4 > x^5$ in this range) So, the only polynomials we have to check are $x^8 - x + 1$ and $x^8 + x^6 - x^3 - x + 1$. The former of these is immediately positive as $1 - x > 0$ when $x \in (0, 1)$. It is also possible to prove that the latter is positive by carefully computing some points (especially around $x = \frac{3}{4}$, close to which the minimum occurs) and checking the derivative around these points. In particular, the interested reader can verify a few facts:

1. At $\frac{3}{4}$, the function evaluates to at least $\frac{1}{10}$.
2. The derivative cannot be less than -2 on $\left[\frac{3}{4}, \frac{4}{5}\right]$ (in fact, it is increasing on this range).
3. The derivative is positive at $\frac{4}{5}$.

These together imply that on the gap of length $\frac{1}{20}$ in $\left[\frac{3}{4}, \frac{4}{5}\right]$, the function could not drop below 0.

Hence, this shows that no such polynomial can have a root in the range $[0, 1]$. Finally, we have to show that no real roots can exist in the range $[1, \infty]$. To do so, we can do a similar elimination procedure to before: it turns out that the only important polynomial is $x^8 - x^7 - x^5 + x^2 + 1$ (odds should push to the left and evens to the right to minimize the derivative, and there should be no crossing). As before, the interested reader can verify the following facts:

1. The derivative cannot be less than -3 on $\left[1, \frac{4}{3}\right]$.
2. The derivative is positive at $\frac{4}{3} - \varepsilon$.

These imply that this polynomial must also be positive on the desired range. Hence, the proof is complete.