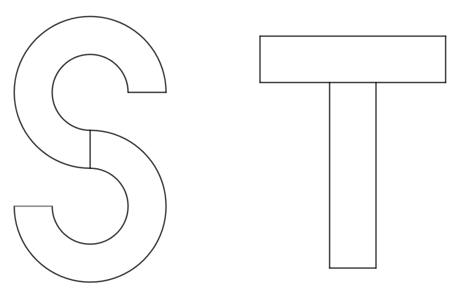
Time limit: 110 minutes.
Instructions: This test contains 25 short answer questions. All answers must be expressed in simplest form unless specified otherwise.
No calculators.

1. Arpit begins with a circular paper disk of radius 8 and cuts out a circular disk of radius 4 with the same center, which results in an annulus. Then, he cuts out and removes a sector of the annulus with a central angle of 90°. He repeats these steps beginning with another circular disk of radius 8 and glues together the two annulus sectors in the shape of an S as shown on the left below. After making the S, Arpit also makes a letter T by gluing together two rectangles each having dimensions $x \times 4x$ as shown on the right below. Compute the value of x Arpit should choose so that the S and T have the same area.



2. Charlie has a large supply of chocolates, each of which has two properties: the filling can be raspberry or blueberry and the shape can be circular or square. Half of the chocolates have a raspberry filling and the other half have a blueberry filling. Also, $\frac{4}{9}$ of the raspberry-filled chocolates are circular and $\frac{2}{3}$ of the blueberry-filled chocolates are square. What fraction of the square chocolates are raspberry-filled?

3. Let p be an odd prime, and P be the *second* smallest multiple of p that is a perfect cube. How many positive factors does P have?

4. There exists a unique real value of x such that

$$(x + \sqrt{x})^2 = 16.$$

Compute x.

5. For all positive integers $n > 1$, let $f(n)$ denote the largest odd proper divisor of n (a proper divisor of n is a positive divisor of n except for n itself). Given that $N = 20^{23} \cdot 23^{20}$, compute

$$\frac{f(N)}{f(f(f(N)))}.$$

6. Let X be the set of natural numbers with 10 digits comprising of only 0's and 1's, and whose first digit is 1. How many numbers in X are divisible by 3?

7. If x and y are positive integers that satisfy $43x + 47y = 2023$, compute the minimum possible value of $x + y$.

8. Lines are drawn from a corner of a square to partition the square into 8 parts with equal areas. Another set of lines is drawn in the same way from an adjacent corner. How many regions are formed inside the square and are bounded by drawn lines and edges of the square?

9. William has a large supply of candy bars and wants to choose one of among three families to give the candy to. Family A has 13 children, family B has 11 children, and family C has 7 children. The children in family C each require an even number of candy bars. If William attempts to distribute the candy bars equally among the children in families A, B, and C, there are 7, 5, and 8 candy bars left over, respectively. What is the least number of candy bars that William could have?

10. Consider the rectangle with a length of 2 and a width of 1. Pick one of the two diagonals of the rectangle. Observe that this diagonal separates the rectangle into two right-angled triangles, R_1 and R_2. Reflect R_1 in the diagonal to obtain R_1'. Compute the area of the intersection of R_1' and R_2.

11. Nathan has discovered a new way to construct chocolate bars, but it's expensive! He starts with a single 1×1 square of chocolate and then adds more rows and columns from there. If his current bar has dimensions $w \times h$ (w columns and h rows), then it costs w^2 dollars to add another row and h^2 dollars to add another column. What is the minimum cost to get his chocolate bar to size 20×20?

12. Let $A_1A_2\ldots A_{12}$ be a regular dodecagon. Equilateral triangles $\triangle A_1A_2B_1, \triangle A_2A_3B_2, \ldots,$ and $\triangle A_{12}A_1B_{12}$ are drawn such that points $B_1, B_2, \ldots,$ and B_{12} lie outside dodecagon $A_1A_2\ldots A_{12}$. Compute the ratio of the area of dodecagon $B_1B_2\ldots B_{12}$ to the area of dodecagon $A_1A_2\ldots A_{12}$.

13. Two frogs jump along a straight line in the same direction, starting at the same place. Every ten seconds, each frog jumps 2, 4, or 6 feet, with each possibility being equally likely. What is the probability that the frogs have traveled the same distance after thirty seconds?

14. How many trailing zeros does the value

$$300 \cdot 305 \cdot 310 \cdots 1090 \cdot 1095 \cdot 1100$$

end with?

15. Let A, B, and C be three points on a line (in that order), and let X and Y be two points on the same side of line AC. If $\triangle AXB \sim \triangle BYC$ and the ratio of the area of quadrilateral $AXYC$ to the area of $\triangle AXB$ is $111 : 1$, compute $\frac{BC}{BA}$.

16. Aidan has five final exams to take during finals week, each on a different weekday. During finals week, there are heavy storms and there is a 48.8% chance of a tree on campus falling down at some point in any given 24-hour period, where the probability of a tree falling down is uniform for the entire week and independent at different instances in time (i.e., a tree falling down at 9 AM does not affect the probability a tree falls down at 9:05 AM). On each day, if a tree falls down at any point between 9 AM and 5 PM, then Aidan's final for that day is canceled. What is the probability that at least two of his finals are canceled?

17. Suppose we have a triangle $\triangle ABC$ with $AB = 12$, $AC = 13$, and $BC = 15$. Let I be the incenter of triangle ABC. We draw a line through I parallel to BC intersecting AB at point D and AC at point E. What is the perimeter of triangle $\triangle ADE$?

18. Ryan chooses five subsets S_1, S_2, S_3, S_4, S_5 of $\{1,2,3,4,5,6,7\}$ such that $|S_1| = 1, |S_2| = 2, |S_3| = 3, |S_4| = 4$, and $|S_5| = 5$. Moreover, for all $1 \leq i < j \leq 5$, either $S_i \cap S_j = S_i$ or $S_i \cap S_j = \emptyset$ (in other words, the intersection of S_i and S_j is either S_i or the empty set). In how many ways can Ryan select the sets?

19. Bernie has an infinite supply of Nerds and Smarties with the property that eating one Nerd increases his IQ by 10 and eating one Smartie increases his IQ by 14. If Bernie currently has an IQ of 99, how many IQ values between 100 and 200, inclusive, can he achieve by eating Nerds and Smarties?

20. If the sum of the real roots x to each of the equations

$$2^{2x} - 2^{x+1} + 1 - \frac{1}{k^2} = 0$$

for $k = 2, 3, \ldots, 2023$ is N, what is 2^N?

21. Equilateral triangle $\triangle ABC$ is inscribed in circle Ω, which has a radius of 1. Let the midpoint of BC be M. Line AM intersects Ω again at point D. Let ω be the circle with diameter MD. Compute the radius of the circle that is tangent to BC on the same side of BC as ω, internally tangent to Ω, and externally tangent to ω.

22. If (a, b) is a point on the circle centered at $(5, 0)$ with radius 4 in the xy-plane, compute the maximum possible value of $\frac{a^2+7b^2}{2a^2+b^2}$.

23. An ant begins walking while facing due east and every second turns $60°$ clockwise or counterclockwise, each with probability $\frac{1}{2}$. After the first turn the ant makes, what is the expected number of turns (not including the first turn) it makes before facing due east again?

24. Equilateral triangle $\triangle ABC$ has side length 12 and equilateral triangles of side lengths $a, b, c < 6$ are each cut from a vertex of $\triangle ABC$, leaving behind an equiangular hexagon $A_1A_2B_1B_2C_1C_2$, where A_1 lies on AC, A_2 lies on AB, and the rest of the vertices are similarly defined. Let A_3 be the midpoint of A_1A_2 and define B_3, C_3 similarly. Let the center of $\triangle ABC$ be O. Note that OA_3, OB_3, OC_3 split the hexagon into three pentagons. If the sum of the areas of the equilateral triangles cut out is $18\sqrt{3}$ and the ratio of the areas of the pentagons is $5:6:7$, what is the value of abc?

25. Consider a sequence $F_0 = 2$, $F_1 = 3$ that has the property $F_{n+1}F_{n-1} - F_n^2 = (-1)^n \cdot 2$. If each term of the sequence can be written in the form $a \cdot r_1^n + b \cdot r_2^n$, what is the positive difference between r_1 and r_2?