1. Let A and B be points $(0, 9)$ and $(16, 3)$ respectively on a Cartesian plane. Let point C be the point $(a, 0)$ on the x-axis such that $AC + CB$ is minimized. What is the value of a?

   Answer: 12

   Solution: Let B' be the reflection of B over the x-axis, i.e. $(16, -3)$. We can try to minimize $AC + CB'$ instead. This occurs when ACB' is a straight line. By similar triangle, we have

   $$\frac{9}{a} = \frac{9+3}{16}.$$

   Hence, $a = \boxed{12}$.

2. William is popping 2022 balloons to celebrate the new year. For each popping round he has two attacks that have the following effects:

   (a) halve the number of balloons (William can not halve an odd number of balloons)

   (b) pop 1 balloon

   How many popping rounds will it take for him to finish off all the balloons in the least amount of moves?

   Answer: 18

   Solution: William can write out 2022 in its binary representation 11111100110. Note that in order reduce the number of balloons when the rightmost digit is 0, William should halve the number of balloons. This takes 1 move to reduce the number of digits by 1. When the rightmost digit is 1, William should pop 1 balloon and then halve the numbers of balloons, which takes 2 moves to reduce the number of digits by 1. However, William does need to halve the balloons afterward if there was only 1 balloon left. Taking these steps into account, we see that there are 8 1's and 3 0's in the binary representation. To get rid of the 1's requires $8 \cdot 2 - 1 = 15$ moves and to get rid of the 0's requires 3 moves, for a total of $\boxed{18}$.

3. What is the numerical value of $(\log_{s^2} m^5)(\log_{m^3} t^6)(\log_{t^5} s^8)$?

   Answer: 8

   Solution: We can rewrite and simplify the expression as follows:
   $\frac{\log m^5}{\log s^2}\frac{\log t^6}{\log m^3}\frac{\log s^8}{\log t^5} = \frac{\log m^5}{\log m^3}\frac{\log t^6}{\log t^5}\frac{\log s^8}{\log s^2} = \log_{m^3} m^5 \log_{t^5} t^6 \log_{s^2} s^8 = \frac{5}{3}\frac{6}{5}\frac{8}{2} = 8$

4. Arpit is hanging Christmas lights on his Christmas tree for the holiday season. He decides to hang 12 rows of lights, but if any row of lights is defective then the Christmas tree will not be lit. If the tree is not lit when he plugs in his lights, how many subsets of rows of lights can be broken for the lights to not work?

   Answer: 4095

   Solution: Since any one of the rows of lights does not work, it could be all the rows, one of the rows, or a few of the rows that does not work. Therefore, we can treat each row as an on and off switch with two states. Since there are 12 rows, we get 2^{12} ways and subtracting the case where all the lights work we get $2^{12} - 1 = \boxed{4095}$.

5. Let $(1 + 2x + 4x^2)^{2020} = a_0 + a_1x + ... + a_{4040}x^{4040}$. Compute the largest exponent k such that 2^k divides

   $$\sum_{n=1}^{2020} a_{2n-1}.$$

Answer: 3

Solution: We note that $(1+2x+4x^2)^{2020} - (1-2x+4x^2)^{2020} = 2(a_1x + a_3x^3 + ...)$. So, we compute know our sum is

$$\frac{7^{2020} - 3^{2020}}{2} = \frac{1}{2} \times (7^{1010} - 3^{1010})(7^{1010} + 3^{1010}).$$

We note that $7 \equiv 3 \equiv -1 \mod 4$. So, $7^{1010} + 3^{1010} \equiv 2 \mod 4$. Next, we simplify

$$7^{1010} - 3^{1010} = (7^{505} - 3^{505})(7^{505} + 3^{505}).$$

We also have that $7^{505} + 3^{505} \equiv 2 \mod 4$. Next, $505 \equiv 1 \mod 4$, so $7^{505} - 3^{505} \equiv 4 \mod 8$. So, the largest power of 2 dividing the sum is $\boxed{3}$.

6. Frank is trying to sort his online friends into groups of sizes n and $n+2$, for some unknown positive integer n, such that each friend is placed into exactly one group; there can be any number of groups of each of the two sizes. He finds that it is impossible to do so with his current number of friends, but would be possible if he had any *even* number of additional friends. If Frank has less than 400 friends, what is the maximum possible number of friends he has currently?

   Answer: 362

   Solution: There are two cases to consider: either n is odd or n is even. If n is odd, then n and $n+2$ are relatively prime, so we can use the Chicken McNugget Theorem to see that Frank would have $n(n+2) - n - (n+2) = n^2 - 2$ friends. The maximum possible value of n is then 19, giving $19^2 - 2 = 359$ friends. If n is even, then n and $n+2$ can be written as $2k$ and $2(k+1)$ for some integer k. The Chicken McNugget Theorem applied to k and $k+1$ gives $k(k+1) - k - (k+1) = k^2 - k - 1$. So, any even number greater than $2(k^2 - k - 1)$ can be expressed using $2k$ and $2(k+1)$. The maximum possible value of k is 14, giving $2(14^2 - 14 - 1) = \boxed{362}$ friends.

7. How many 9-digit numbers are there with unique digits from 1 to 9 such that the first five digits form an increasing series and the last five digits form a decreasing series?

   Answer: 70

   Solution: The middle digit must be greater than the digits to its left and its right, so it must be 9. Now, there are $\binom{8}{4}$ to choose which four digits go before 9 and which four digits go after nine. For each of these possibilities, there is exactly one way to arrange all the digits so that the first four are increasing and the last four are decreasing. Therefore, the answer is $\boxed{70}$.

8. Compute the number of ordered triples (a, b, c) with $0 \leq a, b, c \leq 30$ such that 73 divides $8^a + 8^b + 8^c$.

   Answer: 6600

   Solution: We claim that the set of solutions is exactly the (a, b, c) where all three are different modulo 3.

   As some inspiration, notice that $73 = 8^2 + 8 + 1$: this suggests that we want (as polynomials) $x^2 + x + 1$ dividing $p(x) = x^a + x^b + x^c$. Indeed, since the roots of $x^2 + x + 1$ are ω, ω^2 for ω a third root of unity, we find that $p(x) \equiv x^{a \bmod 3} + x^{b \bmod 3} + x^{c \bmod 3}$, and the only option which works is exactly where the three are different modulo 3.

   Now we show that there are no other solutions. To do so, notice that $8^3 - 1 = (8^2+8+1)(8-1) \equiv 0 \bmod 73$. So, if we assume WLOG that $c \leq b \leq a$, then we require $8^{(a-c) \bmod 3} + 8^{(b-c) \bmod 3} + 1 \equiv$

0 mod 73. However, note that 8^x is one of $1, 8, 64$ mod 73 and the only way for that sum to be 0 is to have $1 + 8 + 64 = 73$. This was already covered by the first case, so there are no other solutions.

Therefore, our answer is $3! \cdot 11 \cdot 10 \cdot 10 = \boxed{6600}$.

9. Mark plays a game with a circle that has six spaces around it, labeled 1 through 6, and a marker. The marker starts on space 1. On each move, Mark flips a coin. If he gets tails, the marker stays where it is, and if he gets heads, he then rolls a die, with numbers 1 through 6, and moves the marker forward the number of spaces that is rolled without stopping (if the marker passes space 6, it will keep going to space 1). What is the expected numbers of moves for the marker to stop on space 6 for the first time?

Answer: 12

Solution: Note that at each space, except for space 6, the probability that the marker moves to space 6 is $\frac{1}{12}$, because there is a $\frac{1}{2}$ chance of getting heads and then a $\frac{1}{6}$ chance of rolling the needed number of steps. The number of expected moves is then the reciprocal of this probability, so we get $\boxed{12}$.

Alternatively, we could set up a system of equations where each equation is similar to $E(1) = \frac{1}{12}(E(2) + E(3) + E(4) + E(5) + E(1)) + \frac{1}{2}E(1) + 1$. Note that $E(6) = 0$. Adding up all five equations of this form (we exclude $E(6)$), and letting $E(1) + \ldots + E(5) = S$, we get $S = \frac{11}{12}S + 5 \Rightarrow S = 60$. Then, we can use the previous equation to see that $E(1) = 12$.

10. Let $P(x) = x^2 + bx + c$ be a polynomial with integer coefficients. Given that $c = 2^m$ for an integer $m < 100$, how many possible values of b are there such that $P(x)$ has integer roots?

Answer: 5100

Solution: Since c is an integer less than 2^{100}, it can equal any of $2^m, 0 \leq m \leq 99$. Let the roots of $P(x)$ be r_1 and r_2. Then $r_1r_2 = c \implies r_1 = 2^{m_1}, r_2 = 2^{m_2}$, where, without loss of generality, $0 \leq m_1 \leq m_2 \leq m$. Since we require $m_1 + m_2 = m$, this yields a total of $\lceil \frac{m+1}{2} \rceil$ pairs (m_1, m_2) for any m. Summing over all our values of m:

$$\sum_{m=0}^{99} \left\lceil \frac{m+1}{2} \right\rceil = 1 + 1 + 2 + 2 + ... + 49 + 49 + 50 + 50$$

$$= 2\sum_{n=1}^{50} n = 50(51) = 2550$$

However, we identify that $-r_1, -r_2$ can also be roots for any given pairing (m_1, m_2), and so our total number of possible values for b is $2 \cdot 2550 = 5100$.

11. A ring of six identical spheres, in which each sphere is tangent to the spheres next to it, is placed on the surface of a larger sphere so that each sphere in the ring is tangent to the larger sphere at six evenly spaced points in a circle. If the radius of the larger sphere is 5, and the circle containing the evenly spaced points has radius 3, what is the radius of each of the identical spheres?

Answer: $\frac{15}{7}$

Solution: Let the radius of the identical spheres be r. If we draw line segments connecting each of the centers of the identical spheres, we get a regular hexagon with side length $2r$, and the distance from a vertex to the center is also $2r$. We consider one of the identical spheres. The center of this sphere, the center of the larger sphere, and the center of the right hexagon form a right triangle with hypotenuse of length $5+r$ and a leg of length $2r$. We can get a triangle similar to this one by connecting the point of tangency between the spheres, the center of the larger sphere, and the center of the circle containing the six points of tangency of the smaller spheres with the larger sphere. This triangle has a hypotenuse of 5 and a leg of length of 3. We can set up the equation $\frac{5+r}{2r} = \frac{5}{3} \Rightarrow 15 + 3r = 10r \Rightarrow r = \boxed{\frac{15}{7}}$.

12. Given a sequence of coin flips, such as 'HTTHTHT...', we define an *inversion* as a switch from H to T or T to H. For instance, the sequence 'HTTHT' has 3 inversions.

    Harrison has a weighted coin that lands on heads $\frac{2}{3}$ of the time and tails $\frac{1}{3}$ of the time. If Harrison flips the coin 10 times, what is the expected number of inversions in the sequence of flips?

    Answer: 4

    Solution: Note that there are 9 possible pairs of flips for an inversion to occur. At each pair of flips, there is a $\frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}$ chance for the two coins to have different orientations. By linearity of expectation, we can sum these 9 equivalent expecations to get $9 \cdot \frac{4}{9} = \boxed{4}$.

13. What is the largest prime factor of $33^4 + 32^4 - 1$?

    Answer: 23

    Solution: Let $x = 32$. The expression becomes $(x+1)^4 + x^4 - 1$. Note that $x = -1$ is a root, so we can factor out $(x+1)$. We have $(x+1)^4 + x^4 - 1 = 2x^4 + 4x^3 + 6x^2 + 4x = 2x(x+1)(x^2+x+2)$. Then, $33^4 + 32^4 - 1 = 2(32)(33)(32^2 + 32 + 2)$. The last factor is $32^2 + 32 + 2 = (33)(32) + 2 = 2((33)(16) + 1) = 2(529) = 2(23)^2$. Thus, the largest prime factor of the number is $\boxed{23}$.

14. Compute

$$\int_e^5 \left(\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x\right) \ln x dx.$$

    Answer: $\left(\frac{5}{e}\right)^5 - \left(\frac{e}{5}\right)^5$

    Solution: We begin with the u-substitution $u = \frac{x}{e}$ to get the integral

$$\int_1^{5/e} (u^{eu} + u^{-eu})(e + e \ln u) du.$$

    Recall that we can compute the derivative of x^x as the derivative of $e^{x \ln x}$. Using the same method, we can find that the derivative of u^{eu} is $u^{eu}(e + e \ln u)$ and the derivative of u^{-eu} is $-u^{-eu}(e + e \ln u)$, so the indefinite integral is $u^{eu} - u^{-eu}$, and evaluating at the endpoints gives the answer.

15. What is the maximum value of x^2y^3 if x and y are non-negative integers satisfying $x + y \leq 9$?

    Answer: 2000

    Solution: First, we realize that the equality must be achieved for x^2y^3 to be maximized. So, we want to maximize $x^2(9-x)^3$. The critical values are achieved at x satisfying $2x(9-x)^3 - 3x^2(9-x)^2 = x(9-x)^2(18-2x-3x) = 0$, i.e. $x = 0, 9, \frac{18}{5}$. When $x = 0$ or $x = 9$, $x^2(9-x)^3 = 0$. The two closest integers to $\frac{18}{5}$ are 3 and 4. When $x = 3$, $x^2(9-x)^3 = 1944$ and when $x = 4$,

$x^2(9-x)^3 = 2000$. Hence, the maximum value of x^2y^3 for non-negative integers x, y is $\boxed{2000}$ when $x = 4$ and $y = 5$.

16. Find the number of ordered triples (a, b, c) such that $a, b, c \in \{1, 2, 3, ..., 100\}$ and a, b, c form a geometric progression in that order.

Answer: 310

Solution: A geometric progression of 3 terms can be written as the form $(m \cdot x^2, m \cdot xy, m \cdot y^2)$ with the conditions that $m, x, y \in \mathbb{N}$ and x and y are relatively prime positive integers. Since any increasing progression can be flipped to form a decreasing progression, we only need to consider the number of static and increasing geometric progressions.

If $y = x$, then the ordered triples are of the form (m, m, m). There are 100 such triples.

Assuming $y > x$ without loss of generality, we know that $m \cdot y^2 \leq 100$. This means that $y \in \{1, 2, 3, ..., 10\}$, and it remains to find the possible values of m and x for each possible value of y. Since $m \cdot y^2 > m \cdot x^2 \geq 1$, the value of m does not depend on the value of x or vice versa. Note that m is bounded by the inequality $m \cdot y^2 \leq 100$ and x is bounded by $x < y$ and x, y must be relatively prime (to avoid double-counting by changing the value m). Summing up the possible values of m and x as y decreases from 10 to 1, there are $4 \cdot 1 + 6 \cdot 1 + 4 \cdot 1 + 6 \cdot 2 + 2 \cdot 2 + 4 \cdot 4 + 2 \cdot 6 + 2 \cdot 11 + 1 \cdot 25 + 0 \cdot 100 = 105$.

If $y < x$, then there are also 105 triples.

Adding these up, there are a total of $100 + 2(105) = 310$ such triples.

17. Compute the number of $1 \leq n \leq 100$ for which $b^n \equiv a \bmod 251$ has a solution for at most half of all $1 \leq a \leq 251$.

Answer: 20

Solution: Let $p = 251$ which is prime. We claim that the answer is exactly the set of $1 \leq n \leq 100$ which have $\gcd(n, p-1) > 2$.

Indeed, since p is prime this implies that we may find a generator g so that $\{1, g, g^2, \ldots, g^{p-2}\}$ are all distinct modulo p. Suppose that $\gcd(n, p-1) = d$. It then follows that $g^{n\frac{p-1}{d}} \equiv 1 \bmod p$, as $d \mid n$ and Fermat's Little Theorem gives $g^{p-1} \equiv 1 \bmod p$. In fact, $\frac{p-1}{d}$ is the smallest $j > 0$ for which $g^{n \cdot j} \equiv 1 \bmod p$.

So, $g^{n \cdot j} \equiv g^{n \cdot (j + k\frac{p-1}{d})}$, implying that b^n can only achieve at most $\frac{p-1}{d}$ many values. In fact, it achieves exactly this many values: suppose that $g^{n \cdot j} \equiv g^{n \cdot k} \bmod p$ when $0 \leq j < k < \frac{p-1}{d}$. Then, $g^{n \cdot (j-k)} \equiv 1 \bmod p$. Therefore, we need either $j = k$ or $\frac{p-1}{d} \mid j - k$, both impossible.

Hence, it suffices to have $\gcd(n, p-1) > 2$. Now, we simply count $1 \leq n \leq 100$ divisible by 5 (as $p - 1 = 250$), of which there are $\boxed{20}$.

18. Let $f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + 16$ be a polynomial with nonnegative real roots. If $(x-2)(x-3)f(x) = x^6 + b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + 96$, what is the minimum possible value of b_2?

Answer: 320

Solution: Let r_1, r_2, r_3, and r_4 be the roots of $f(x)$. By Vieta's formulas, we have $r_1r_2r_3r_4 = 16$. When $(x-2)(x-3)f(x)$ is expanded, we see that

$$\begin{aligned} b_2 &= 6a_2 - 5a_1 + 16 \\ &= 6(r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4) + 5(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4) + 16. \end{aligned}$$

By the AM-GM inequality, we have

$$r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 \geq 6\sqrt{r_1r_2r_3r_4} = 24$$

and

$$r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4 \geq 4\sqrt[4]{r_1^3r_2^3r_3^3r_4^3} = 32$$

with equality when $r_1 = r_2 = r_3 = r_4 = 2$. So, the minimum possible value of b_2 is $6 \cdot 24 + 5 \cdot 32 + 16 = \boxed{320}$.

19. Define a *brook* as a chess piece which can move to any square which is exactly 2 positions away. Specifically, a brook at position (x, y) can move to any (x', y') with $|x' - x| + |y' - y| = 2$. What is the maximum number of brooks that can be placed on a 6×6 chessboard so that no two attack each other?

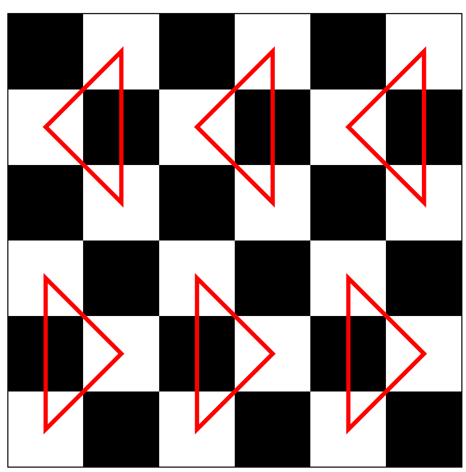
Answer: 12

Solution: Let's rephrase this as a graph problem. Consider the graph where each position (x, y) is adjacent to the (x', y') satisfying these conditions. Then, we are searching for a maximum independent set (set of vertices so that no two share an edge).

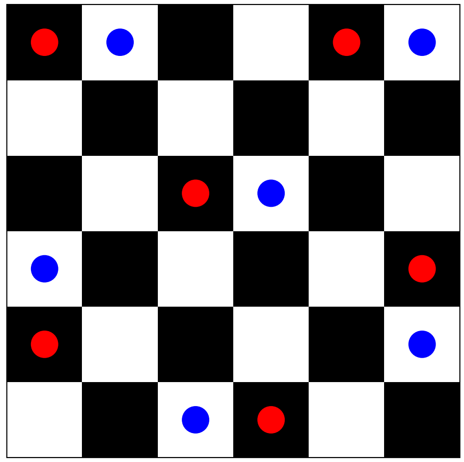
To this end, define a vertex cover in a graph G as a set of vertices S so that every edge in G has at least one endpoint in S. The claim is that if G has n vertices and a minimum vertex cover of size $|S|$, then the maximum independent set has size exactly $n - |S|$. Indeed, let's take all the vertices not in S to be the independent set. Then, there cannot be an edge between any pair of them, else S is not a vertex cover. Similarly, we can transform back from an independent set to a vertex cover. The punchline is that if we show that $|S| \geq k$ on our chessboard graph, then it follows that the maximum number of brooks is at most $36 - k$.

Notice that the black and white squares on the chessboard are entirely disjoint (there are no edges between them) and they are symmetric, so we may solve the problem on each separately and double the final answer.

Consider the following crucial fact: if H is a triangle, then the minimum vertex cover has size 2. Indeed, as each pair of vertices has an edge between them, we cannot have two vertices not in the vertex cover. So, consider deconstructing our graph into 6 disjoint triangles:



This implies that the minimum vertex cover has size at least $6 \cdot 2 = 12$. Therefore, gives a bound on the maximum of $36 - 2 \cdot 12 = \boxed{12}$ which is achievable by the following construction.



20. Determine the number of pairs (x, y) where $1 \leq x, y \leq 2021$ satisfying the relation

$$x^3 + 21x^2 + 484x + 6 \equiv y^2 \pmod{2022}.$$

Answer: 2018

Solution: First, let us extend our allowed bounds to $0 \leq x, y \leq 2021$ and then eliminate the cases that use 0. From the Chinese Remainder theorem, we can consider the problem phrased in primes, and then multiply our results. Let us first look at the problem in modulo 2. It reduces to

$$x^3 + x^2 = y^2 \pmod 2$$

which is satisfied for $(x, y) \to (0, 0), (1, 0)$. In modulo 3 we get the expression

$$x^3 + x \equiv y^2 \pmod 3$$

which gives us that $(x, y) \equiv (2, 1), (2, 2), (0, 0) \pmod 3$.

Finally, looking in mod 337 we get the expression

$$x^3 + 21x^2 + 147 + 6 \equiv y^2 \pmod{337}$$

but, since $343 \equiv 6 \pmod{337}$ this reduces to $(x + 7)^3 \equiv y^2 \pmod{337}$, so it is equivalent to finding triples (x, y, z) such that $x^3 \equiv y^2 \equiv z \pmod{337}$, of which we claim there are 337. Note that z must be both a cubic residue and a quadratic residue. Because $337 \equiv 1 \pmod 3$, it follows that there are $\frac{p-1}{3}$ cubic residues, and half of these are quadratic residues as well, and so there are $\frac{p-1}{6}$ possible values for z. But, for each z, there are 2 choices for y and 3 choices for x, and so we have that there are 6 pairs (x, y) for each z. Adding the trivial case of $x = y = z = 0$ we have that there are 337 such triples.

Then we get that there are $2 \cdot 3 \cdot 337 = 2022$ pairs (x, y) that satisfy this equivalence in the range $0 \leq x, y \leq 2021$. We must subtract the cases when x or y is 0. Suppose that x is 0, then we must have $(x, y) \to (0, 0) \pmod{2, 3}$ and $x \equiv 0 \pmod{337}$ while $y^2 \equiv 343 \pmod{337}$, which can quickly be confirmed to be a quadratic residue by quadratic reciprocity, noting that $\left(\frac{343}{337}\right) = \left(\frac{7}{337}\right)^3 = (-1)\left(\frac{337}{7}\right)^3 = (-1)\left(\frac{6}{7}\right)^3 = 1$, since 6 is not a quadratic residue mod 7. This gives a count of 2 for the case of $x = 0$. When $y = 0$, we have $(x, y) \to (330, 0) \pmod{337}, (0, 0) \pmod 3$ and then we can have either $(0, 0)$ or $(1, 0)$ in mod 2 which is another count of 2. Thus our final answer is

$$2022 - 2 - 2 = \boxed{2018}.$$

21. Let $\triangle ABC$ be an acute triangle with orthocenter H, circumcenter O, and circumcircle Γ. Let the midpoint of minor arc BC on Γ be M. Suppose that $AHMO$ is a rhombus. If BH and MO intersect on segment AC, determine
$$\frac{[AHMO]}{[ABC]}.$$

Answer: $\mathbf{1-\frac{\sqrt{3}}{3}}$ **or** $\mathbf{\frac{3-\sqrt{3}}{3}}$

Solution: We must have $AH = AO = R$, the radius of Γ. Let D and E be the feet of the altitudes from B and C to AC and AB, respectively. We will denote $\angle BAC$ as $\angle A$. Note that $\triangle ABC$ and $\triangle ADE$ are homothetic with ratio $\cos A$. Furthermore, note that AH is the diameter of (ADE), so $AH = \frac{DE}{\sin A} = \frac{BC\cos A}{\sin A}$. Also, $AO = R = \frac{BC}{2\sin A}$, so we have
$$\frac{\cos A}{\sin A} = \frac{1}{2\sin A}$$
so $\cos A = \frac{1}{2}$ and $\angle BAC = 60°$. Now because MO, the perpendicular bisector of BC, intersects D, $BD = CD$, so BDC is a right isosceles triangle, so $\angle BCA = 45°$. It also follows from simple angle chasing that $\angle OAH = 30°$. Now we have that
$$[ABC] = \frac{BC^2 \sin B \sin C}{2\sin A} = \frac{BC^2 \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{6}+\sqrt{2}}{4}}{\sqrt{3}} = \frac{BC^2(\sqrt{3}+1)}{4\sqrt{3}}$$
and
$$[AHMO] = \left(\frac{BC}{\sqrt{3}}\right)^2 \sin 30 = \frac{BC^2}{6}$$
so our answer is
$$\frac{4\sqrt{3}}{6(\sqrt{3}+1)} = \boxed{\frac{3-\sqrt{3}}{3}}$$

22. *Note: this round consists of a cycle, where each answer is the input into the next problem.*

Let k be the answer to problem 24. Ariana Grande has k identical rings on a table, each of radius 1. She wishes to split these rings into necklaces, but with the added constraint that two adjacent rings on a necklace have to subtend an arc of at least 145°. How many ways are there to partition these rings into circular necklaces so that no two unlinked rings intersect or are tangent? Note that two necklaces of the same number of rings are seen as identical.

An alternative and equivalent formulation of the problem: In how many ways can you partition a number k into unordered tuples $(k_1, k_2, \dots)$ such that $k_1 + k_2 + \cdots = k$ with each $k_i \geq 11$?

Answer: 10

Solution: Consider the centers of the rings, and notice that if we have a linked n-gon, then one of the angles is at most $180\left(1 - \frac{2}{n}\right)$. Then, we need this angle to be greater than $a°$ (since we have half an arc from each of the circles joined to it). Therefore, we need $1 - \frac{2}{n} > \frac{a}{180} \implies n > \frac{360}{180-a}$. Plugging in $a = 145$ gives that $n \geq 11$.

Now, let's combine the results of the other two problems.

Recall that if the answer to this problem is r, then the answer to the following problem is $3r+6$, and doing another chaining gives an answer of $3r+5$. Hence, we are looking for $k = 3r+5$, and an answer of r.

Suppose that r was big (specifically, we will rule out $r > 14$). Then, let's count the number of ways Ariana can have at most two necklaces. This is $1 + \lfloor \frac{k}{2} \rfloor - 10$ (the smaller necklace has size between 11 and $\lfloor \frac{k}{2} \rfloor$, inclusive). Since $\lfloor \frac{k}{2} \rfloor \geq \frac{3r+4}{2}$, there would be at least $\frac{3r}{2} - 7$ ways for Ariana to partition her rings into two necklaces. But, $\frac{3r}{2} - 7 > r$ as $r > 14$.

On the other hand, if $r < 6$ then Ariana only has at most 20 rings, so she can only make one necklace total. So, we know that our solution satisfies $6 \leq r \leq 14$.

Next, notice that unless $r > 9$, $k \leq 32$ so Ariana can only make at most two necklaces. As discussed before, this only leaves $1 + \lfloor \frac{32}{2} \rfloor - 10 = 7$ possibilities for her splitting her necklaces.

Therefore, $10 \leq r \leq 14$. At this point, it's possible to just do out the computation: $r = 10$ adds exactly two ways of making 3 necklaces, so that's our answer. We can check that $r = 11$ onward has $> r$ possibilities as well.

23. Let r be the answer to problem 22. Let ω_1 and ω_2 be circles of each of radius r, respectively. Suppose that their centers are also separated by distance r, and the points of intersection of ω_1, ω_2 are A and B. For each point C in space, let $f(C)$ be the the incenter of the triangle ABC. As the point C rotates around the circumference of ω_1, let S be the length of the curve that $f(C)$ traces out. If S can be written in the form $\frac{a+b\sqrt{c}}{d}\pi$ for a, b, c, d nonnegative integers with c squarefree and $\gcd(a, b, d) = 1$, then compute $a + b + c + d$.

Answer: 36

Solution: Fix a point C and let the incenter be I. Suppose that L is the midpoint of the minor arc AB, and L' the midpoint of major arc AB. Then, by Fact 5 (otherwise known as the Incenter/Excenter Lemma), L is the center of the circle containing A, B, I. Therefore, this implies that as C rotates, I rotates about this circle (as A, B, L are fixed by construction).

When C is on the ω_1 side of AB, the distance that I covers is $|LA| \cdot \angle ALB$, where the angle is measured in radians.

By symmetry, the distance from L to line AB is $r - \frac{d}{2}$ (call d the distance between centers for now so that they aren't confused for each other), and the length of AB is $2\sqrt{r^2 - \frac{d^2}{4}}$. So, $|LA| = \sqrt{r(2r-d)}$. From here, Law of Cosines gives that $\angle ALB = \cos^{-1}\left(-\frac{d}{2r}\right)$.

When C is on the ω_2 side of AB, note that $L'ALB$ is a cyclic quadrilateral so $\angle AL'B = \pi - \angle ALB$. The Pythagorean theorem yields that $|L'A| = \sqrt{r(2r+d)}$. So, our final answer is

$$\sqrt{r(2r-d)}\cos^{-1}\left(-\frac{d}{2r}\right) + \sqrt{r(2r+d)}\left[\pi - \cos^{-1}\left(-\frac{d}{2r}\right)\right]$$

Plugging in $r = d$ gives that we have $r\cos^{-1}(-\frac{1}{2}) + r\sqrt{3}[\pi - \cos^{-1}(-\frac{1}{2})]$. As $\cos^{-1}(-\frac{1}{2}) = \frac{2\pi}{3}$, we may simplify this to $\frac{2r+r\sqrt{3}}{3}\pi$. This yields an answer of $3r + 6$.

24. Let m be the answer to problem 23. Suppose that $x_1, x_2, \ldots, x_{m-1}$ are each chosen randomly and independently from the set $\{1, 2, \ldots, m\}$. Then, let e_n be the expected value of $\sqrt[n]{\sum_{i=1}^{m-1} x_i^n}$. Compute $\left\lfloor \lim_{n\to\infty} e_n \right\rfloor$.

Answer: 35

Solution: We claim that $\lim_{n\to\infty} e_n = \max\{x_1, x_2, \ldots, x_{m-1}\}$. WLOG suppose $x_1 \geq x_2, x_3, \ldots, x_{m-1}$. Then, $e_n \geq x_1$ since $x_i^n \geq 0$ for all i. Furthermore, $e_n \leq \sqrt[n]{mx_1^n} = x_1\sqrt[n]{m}$, and as $\sqrt[n]{m} \to 1$ when $n \to \infty$, this implies that $x_1 \leq \lim_{n\to\infty} e_n \leq x_1$.

Hence, it suffices to compute the expectation of $\max\{x_1, x_2, \ldots, x_{m-1}\}$.

In particular, we claim that this is between $m-1$ and m (exclusive on both bounds), yielding an answer of $m-1$. There are many ways to show this (explicitly bounding the expectation is possible, see the end of this solution for an informal discussion of how to do that), but here we'll highlight a somewhat "elegant" method called *coupling* of random variables. The idea of coupling for lower bounds is showing that there is a different random variable which always takes a smaller value, whose expectation is easier to compute. Then, we immediately get a lower bound on the expectation of our variable.

To begin, note that certainly the maximum of the x_i must be at most m by definition of their choice (and since it's possible for none of them to equal m, this is an exclusive bound). So, all we have to show is the lower bound $m-1$.

Consider choosing $y_1, y_2, \ldots, y_{m-1}$ independently and at random uniformly from the interval $[0, m]$: that is, now y_i need not take integer values. Now, let $x_i' = \lceil y_i \rceil$: round y_i up to the nearest integer. Then, the probability that $x_i' = j$ is equal to $\frac{1}{m}$: this is the probability that y_i is in the interval $(j-1, j]$.

So, we have that $x_i' \geq y_i$ for each i, so $\max\{x_1', \ldots, x_{m-1}'\} \geq \max\{y_1, \ldots, y_{m-1}\}$. But, x_i' has the same distribution as x_i ! So, if we compute the expectation of the maximum of the y_i, this will give us a lower bound on the expectation of x_i.

Let $f(t)$ be the probability that $\max\{y_1, y_2, \ldots, y_{m-1}\} \leq t$. Then, since the y_i are independent, this is exactly $f(t) = \left(\frac{t}{m}\right)^{m-1}$. Define also $p(t)$ to be the "probability" that this maximum is exactly t. One way to think about this is writing

$$p(t) = \lim_{\delta \to 0} \frac{\text{probability max in } [t, t+\delta]}{\delta},$$

in other words as the "fraction" of times the maximum is infinitesimally close to t.

However, note that the numerator of the above is related to $f(t)$, so

$$p(t) = \lim_{\delta \to 0} \frac{f(t+\delta) - f(t)}{\delta} = f'(t).$$

Now, similar to regular expectation, we have the expectation of the maximum of these variables is

$$\int_0^m tp(t)\,\mathrm{d}t = \frac{1}{m^{m-1}} \int_0^m t \cdot (m-1)t^{m-2}\,\mathrm{d}t = \frac{m-1}{m^{m-1}} \int_0^m t^{m-1}\,\mathrm{d}t = \frac{m-1}{m} \left[\left. \frac{t^m}{m} \right|_{t=0}^{m} \right] = m-1$$

so indeed we have proven that the expectation of $\max\{x_1, x_2, \ldots, x_{m-1}\} \geq m-1$ and we are done.

Now we sketch the argument without coupling, as promised above. Consider the probability that the maximum of the x_i is at most t: this is $g(t) = \left(\frac{t}{m}\right)^{m-1}$.

It is a fact about expectations of a random variable that the expectation is equal to $\sum_{t=0}^{\infty} 1 - g(t)$ (and can be proved by telescoping the definition of expectation), so in our case this yields

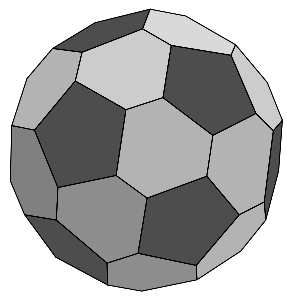
$$\sum_{t=0}^{m-1} 1 - \left(\frac{t}{m}\right)^{m-1} = m - \frac{1}{m^{m-1}} \sum_{t=0}^{m-1} y^{m-1}.$$

We prove that $\sum_{t=0}^{m-1} y^{m-1} \leq m^{m-1}$, which will give the conclusion. Notice that

$$\sum_{y=0}^{m-1} y^{m-1} \leq \int_0^m y^{m-1}\,\mathrm{d}y = m^{m-1}$$

(similar reasoning as the coupling) so we are done as before.

25. A convex regular icosahedron has 20 faces that are all congruent equilateral triangles, and five faces meet at each vertex. A regular pentagonal pyramid is sliced off at each vertex so that the vertex lies directly above the center of the base (the diagram shows an example of an icosahedron with the pyramids sliced off, for some arbitrary size of the pyramids). The icosahedron has edge length 1. If the sliced off pyramids are identical and do not overlap, what is their largest possible total volume?



Answer: $\mathbf{\frac{5+\sqrt{5}}{16}}$

Solution: Note that the triangular faces of each pentagonal pyramid are equilateral triangles, so all of the edges of the pyramid have the same length. This means two edge lengths are removed from each of the edges of the icosahedron, so the edge length of the pyramids can be at most $\frac{1}{2}$. Also, the icosahedron has 12 vertices since the 20 faces have $20 \cdot 3 = 60$ vertices total, counting each vertex 5 times. It remains to find the volume of one of the pentagonal pyramid. For ease of calculation, we will first assume the edges of the pyramid have length 1 and then scale by a factor of $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

We first find the value of $\sin^2 36°$ and $\sin^2 72°$. For $\sin^2 36°$, we can draw a 36-36-108 triangle and divide the 108° angle into 72° and 36° angles. Then, note that we have formed another 36-36-108 triangle inside the original triangle. Using the similar triangles, we find that the ratio r of the longer side length to the shorter side length of a 36-36-108 triangle satisfies $r^2 = r+1$. This gives the golden ratio, $r = \frac{1+\sqrt{5}}{2}$. Then, we can see that $\cos 36° = \frac{1+\sqrt{5}}{4}$, so $\sin^2 36° = 1 - \cos^2 36° = \frac{5-\sqrt{5}}{8}$. Using the double angle formula, we have $\sin^2 72° = (2 \sin 36° \cos 36°)^2 = \frac{5+\sqrt{5}}{8}$. (This is only one possible way to find these values.)

To find the area of a regular pentagon with side length 1, we divide the pentagon into 5 congruent triangles from its center. Dividing one of these triangles by the altitude to the side of length 1 allows us to find that the distance from the center of the pentagon to a vertex is $\frac{1}{2\sin 36°}$. Then, using Law of Sines on the triangle and multiplying by 5 gives us the area of the pentagon as $5 \cdot \frac{1}{2} \cdot \frac{1}{4\sin^2 36°} \cdot \sin 72° = \frac{5}{8} \cdot \frac{\sin 72°}{\sin^2 36°}$.

In the pyramid, we drop the altitude from the vertex above the base and note that we can form a right triangle with hypotenuse 1 and leg $\frac{1}{2\sin 36°}$, with the other leg being the altitude. We use the Pythagorean theorem to find that the height of the pyramid is $\sqrt{1 - \frac{1}{4\sin^2 36°}}$.

Finally, we can find that the volume of the pyramid is

$$\begin{aligned}
\frac{1}{3}(\text{base})(\text{height}) &= \frac{1}{3}\cdot\frac{5}{8}\cdot\frac{\sin 72^\circ}{\sin^2 36^\circ}\cdot\sqrt{1-\frac{1}{4\sin^2 36^\circ}} \\
&= \frac{5}{24}\cdot\frac{1}{\sin^2 36^\circ}\cdot\sqrt{\sin^2 72^\circ-\frac{\sin^2 72^\circ}{4\sin^2 36^\circ}} \\
&= \frac{5}{24}\cdot\frac{1}{\sin^2 36^\circ}\cdot\sqrt{\sin^2 72^\circ-\cos^2 36^\circ} \\
&= \frac{5}{24}\cdot\frac{1}{\sin^2 36^\circ}\cdot\sqrt{\sin^2 72^\circ+\sin^2 36^\circ-1} \\
&= \frac{5}{24}\cdot\frac{8}{5-\sqrt{5}}\cdot\sqrt{\frac{10}{8}-1} \\
&= \frac{5}{6(5-\sqrt{5})} \\
&= \frac{5+\sqrt{5}}{24}.
\end{aligned}$$

We multiply by the scale factor of $\frac{1}{8}$ and 12 for the number of vertices to get $\frac{12}{8}\cdot\frac{5+\sqrt{5}}{24} = \boxed{\frac{5+\sqrt{5}}{16}}$.

26. Consider the equation

$$\frac{a^2+ab+b^2}{ab-1}=k,$$

where $k\in\mathbb{N}$. Find the sum of all values of k, such that the equation has solutions $a,b\in\mathbb{N}, a>1, b>1$.

Answer: 11

Solution: We can rewrite the equation as

$$a^2+b^2-(k-1)ab=-k \tag{1}$$

We will now use Vieta jumping. Let (a, b) be the minimum sum solution of a fixed k, and, without loss of generality, let $a\ge b$. From the Vieta formulas, we have that $((k-1)b-a,b)$ is also a solution. Substituting in (1), we get

$$\begin{gathered}
b^2=(k-1)ab-a^2-k \\
b^2=(k-1)((k-1)b-a)b-((k-1)b-a)^2-k \\
b^2=-k+a((k-1)b-a)<a((k-1)b-a)
\end{gathered}$$

This implies that $(k-1)b-a>\frac{b^2}{a}>0$, hence it is a valid solution. Since (a,b) is minimal, we get that $b(k-1)-a>a\Leftrightarrow b(k-1)>2a$. By the AM-GM inequality,

$$\frac{a^2+ab+b^2}{3}\ge ab \Longrightarrow \frac{a^2+ab+b^2}{ab-1}>\frac{a^2+ab+b^2}{ab}\ge 3$$

and hence, $k>3$.

Let us now consider (1) as a quadratic with respect to b. Computing the determinant, we get $D = a^2(k-1)^2 - 4a^2 - 4k = a^2(k^2-2k-3) - 4k$. Since b is from the minimum sum solution,

$$b = \frac{(k-1)a - \sqrt{D}}{2}, \quad b(k-1) > 2a \Longrightarrow \frac{(k-1)a - \sqrt{D}}{2} > \frac{2a}{k-1}.$$

Substituting for D and multiplying by $2(k-1)$,

$$\begin{gathered}(k^2-2k-3)a \geq (k-1)\sqrt{a^2(k^2-2k-3) - 4k} \\ (k^2-2k-3)^2a^2 \geq (k-1)^2(a^2(k^2-2k-3) - 4k) \\ a^2(k^2-2k-3) \leq k(k-1)^2\end{gathered}$$

If $a \geq b > 2$, the following inequalities hold:

$$\begin{gathered}ab - 1 \geq 3a - 1 \geq 2a \\ \Rightarrow k = \frac{a^2+ab+b^2}{ab-1} \leq \frac{a^2+ab+b^2}{2a} \leq \frac{3a^2}{2a} = \frac{3a}{2} \\ \Rightarrow \frac{2}{3}k \leq a \Rightarrow k(k-1)^2 \geq \frac{4k^2}{9}(k^2-2k-3) \\ 4k^3 - 17k^2 + 6k - 9 \leq 0\end{gathered}$$

From the last equation, it can easily derived that $k < 5$, since for $k \geq 5$ the value is always positive (evaluate first derivative). Therefore the only possible value for k is 4.

Now considering the case $b = 2$, we have that $a^2 + a(2-2k) + 4 + k = 0$, from which we get that $\sqrt{(2-2k)^2 - 16 - 4k} = \sqrt{k^2-3k-3}$ must be an integer. The square root is bounded above by $k-1$, so we have $k^2 - 3k - 3 = (k-r)^2$ for $r > 1$. Simplifying, we get

$$k = \frac{r^2+3}{2r-3} = \frac{1}{2}(r + 1 + \frac{r+9}{2r-3}),$$

therefore $r + 9 \geq 2r - 3$, hence $r \leq 12$. Evaluating, we get solutions for $r = 2, 3, 5, 12$ and the unique values for k are $k = 4, 7$. Henceforth, the sum of all k is 11.

27. Compute

$$\sum_{n=0}^{1011} \frac{\binom{2022-n}{n}(-1)^n}{2021-n}.$$

Answer: $\mathbf{\frac{4041}{4082420}}$

Solution 1: We note that

$$\sum_{n=0}^{1011} \frac{\binom{2022-n}{n}(-1)^n}{2021-n} = \sum_{n=0}^{1011} \frac{\binom{2021-n}{n} + \binom{2021-n}{n-1}}{2021-n}(-1)^n = \sum_{n=0}^{1010} \frac{\binom{2021-n}{n}}{2021-n}(-1)^n - \sum_{k=0}^{1010} \frac{\binom{2020-k}{k}}{2020-k}(-1)^k$$

by Pascal's Identity. We will need the following claim (that can be motivated by taking small cases) to complete the problem:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\binom{n-k}{k}(-1)^k}{n-k} = \frac{2\cos\frac{n\pi}{3}}{n}$$

This is due to the fact that the nth Chebyshev polynomial has the form

$$T_n(x) = \frac{n}{2}\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^i \frac{\binom{n-i}{i}}{n-i}(2x)^{n-2i}$$

where, we are taking $T_n(\cos\frac{\pi}{3}) = \cos\frac{n\pi}{3}$, hence our sum has the value $\frac{2\cos\frac{n\pi}{3}}{n}$. So the answer to the question at hand is

$$\frac{2\cos\frac{2021\pi}{3}}{2021} - \frac{2\cos\frac{2020\pi}{3}}{2020} = \frac{1}{2021} + \frac{1}{2020} = \boxed{\frac{4041}{4082420}}.$$

Solution 2:

Extend the limits to ∞ instead of 1011, and set $x = -1$. Let the original sum be $f(x)$.
Then, since $\frac{1}{m}\binom{m}{k} = \frac{1}{k}\binom{m-1}{k-1}$, we have

$$\frac{\binom{2022-n}{n}}{2021-n} = \frac{1}{n}\binom{2020-(n-1)}{n-1} + \frac{1}{n-1}\binom{2019-(n-2)}{n-2}.$$

So, we have

$$f(x) = \sum_{n=1}^{\infty}\frac{1}{n}\binom{2020-(n-1)}{n-1}x^n + x\sum_{n=2}^{\infty}\frac{1}{n-1}\binom{2019-(n-2)}{n-2}x^{n-1}$$

and letting the first term be $g(x)$,

$$g'(x) = \sum_{n=0}^{\infty}\binom{2020-n}{n}x^n.$$

Let $h(m) = \sum_{n=0}^{\infty}\binom{m-n}{n}x^n$ so $h(m) = h(m-1) + x\cdot h(m-2)$ with base cases $h(0) = 0, h(1) = 1$.
Now, consider the characteristic function $h(m) = a(x)^m$. Then, $a(x)^2 = a(x) + x$, solving to $a(x) = \frac{1\pm\sqrt{1+4x}}{2}$. Therefore,

$$h(m) = \lambda_1\left(\frac{1+\sqrt{1+4x}}{2}\right)^m + \lambda_2\left(\frac{1-\sqrt{1+4x}}{2}\right)^m.$$

By the initial conditions, $\lambda_1+\lambda_2 = 0$ and $\sqrt{1+4x}(\lambda_1-\lambda_2) = 2$ so $\lambda_1 = \frac{1}{\sqrt{1+4x}}$ and $\lambda_2 = -\frac{1}{\sqrt{1+4x}}$.
Now, note that

$$\begin{aligned} g(m,x) &= \int \frac{1}{\sqrt{1+4x}}\left(\frac{1+\sqrt{1+4x}}{2}\right)^m - \frac{1}{\sqrt{1+4x}}\left(\frac{1-\sqrt{1+4x}}{2}\right)^m \mathrm{d}x \\ &= \frac{1}{(m+1)2^{m+1}}\left[(1+\sqrt{1+4x})^{m+1} + (1-\sqrt{1+4x})^{m+1}\right]. \end{aligned}$$

Hence, $f(x) = g(2020, x) + xg(2019, x)$.
Putting in $x = -1$ means that we are looking for $(1+i\sqrt{3})^{m+1} + (1-i\sqrt{3})^{m+1}$. Since $1+i\sqrt{3} = 2e^{i\frac{\pi}{3}}$, this value is $2^{m+2}\cos\left(\frac{(m+1)\pi}{3}\right)$.
Hence,

$$f(x) = \frac{2\cos\left(\frac{2021\pi}{3}\right)}{2021} - \frac{2\cos\left(\frac{2020\pi}{3}\right)}{2020} = \frac{1}{2021} + \frac{1}{2020} = \boxed{\frac{4041}{4082420}}.$$