Time limit: 110 minutes.
Instructions: This test contains 25 short answer questions. All answers must be expressed in simplest form unless specified otherwise.
No calculators.

1. There are natural numbers a and b, where b is square-free, for which we can write
$$20+\frac{1}{20+\frac{1}{20+\frac{1}{20+\cdots}}}=a+\sqrt{b}.$$
What is $a+\sqrt{b}$?

2. Suppose there are five cars and three roads ahead. Each car selects a road to drive on uniformly at random. Every car adds a one minute delay to the car behind them. What is the expected delay of a car selected uniformly at random from the five cars? (For example, if cars $1, 2, 3$ go on road A in that order and cars $4, 5$ go on road B, then the delays for the cars are $0, 1, 2, 0, 1$ respectively.)

3. Compute
$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\cdots+\frac{1}{17\cdot 19}.$$

4. Consider an acute angled triangle $\triangle ABC$ with sides of length a, b, c. Let D, E, F be points (distinct from A, B, C) on the circumcircle of $\triangle ABC$ such that: $AD \perp BC$, $BE \perp AC$, $CF \perp AB$. What is the ratio of the area of the hexagon $AECDBF$ to the area of the triangle $\triangle ABC$?

5. Let $ABCD$ be a trapezoid with AB parallel to CD and $AB = BC = AD = 8$. Side CD is extended past D to a point E so that $DE = 8$ and $CD = AE$. What is the length of CD?

6. What are the last two digits of $2022^{2022^{2022}}$?

7. On December 9, 2004, Tracy McGrady scored 13 points in 33 seconds to beat the San Antonio Spurs. Given that McGrady never misses and that each shot made counts for 2, 3, or 4 points, how many shot sequences could McGrady have taken to achieve such a feat assuming that order matters?

8. Stanford has a new admissions process that it would like to test out on the Stanford Class of 2027. An admissions officer starts by ordering applicants $1, 2, \ldots,$ and 2022 in a circle with applicant 1 being after applicant 2022. Then, starting with applicant 1, the admissions officer removes every 2023rd applicant. What is the number of the applicant removed in the 49th iteration?

9. Over all pairs of real numbers (x, y) with $x^2+y^2 = 1$, let m be the maximum value of $4xy-8xy^3$. At what values of x is m attained? (List your answers separated by commas, in any order.)

10. You need to bike to class but don't know where you parked your bike. There are two bike racks, A and B. There is a $1/5$ chance for your bike to be at A; it takes one minute to walk to A and four minutes to bike from A to class. Then, there is a $4/5$ chance for your bike to be at B; it takes three minutes to walk to B and five minutes to bike from B to class. However, if your choice is wrong, you need to walk from your original choice A or B to the other, which takes four minutes, before departing to class from there.

Suppose you only care about getting to class on time. For a some interval of minutes before class, going to bike rack B first gives a strictly higher chance of making it to class on time. How many minutes long is that interval (i.e. an interval of 15 minutes before class to 21 minutes before class has length 6)?

11. Suppose that x and y are complex numbers satisfying the relations

$$x^2 + y^2 = 11$$
$$x^3 + y^3 = 20$$
$$x^4 + y^4 = 23$$
$$x^5 + y^5 = -25.$$

Compute $x^6 + y^6$.

12. Let $S_n = \sum_{j=1}^{n} j^3$. Find the smallest positive integer n such that the last three digits of S_n are all zero.

13. Consider right triangles with two legs on the x and y axes with hypotenuse tangent to the line $y = 2022/x$ (the slope of this line at $x = a$ is $-2022/a^2$). If two tangent points are chosen uniformly at random on the curve $y = 2022/x$ restricted to $1/2022 \leq x \leq 2022$, what is the expected ratio of the area of the triangle with larger y-intercept to the area of the triangle with lesser y-intercept?

14. A tiebreaker in tennis is played until a player has seven or more points and is winning by at least two points. Players A and B take turns serving 2 points each in the order ABBAABBAA.... Whoever serves a point has a 70% chance of winning that point. Given that A served first and that the score is currently 4-4, what is the probability that A wins the tiebreaker?

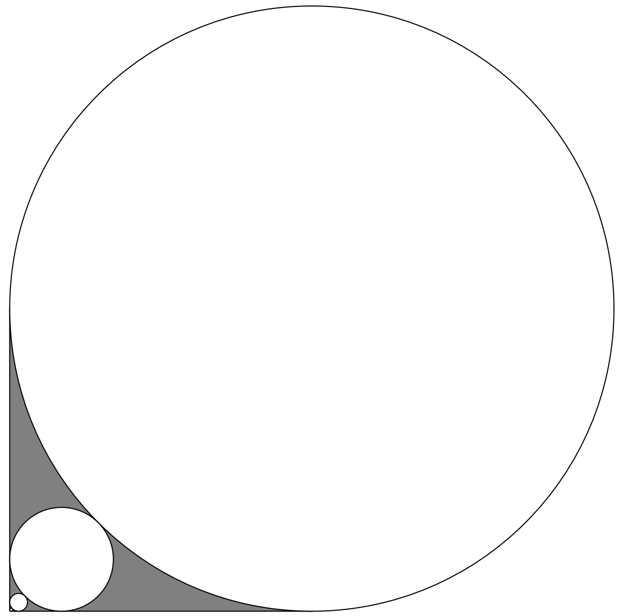
15. What is the unique perfect cube c of the form $c = k^3 + k^2 + 11k + 1$ for some strictly positive integer k?

16. Consider the parabola $y = x^2$. Let a circle centered at $(0, a)$ be tangent to the parabola at (b, c) such that $a + c = (2b - 1)(2b + 1)$. If $a > 0$, find the area of the finite region between the parabola and the circle.

17. Three cities X, Y and Z lie on a plane with coordinates (0, 0), (200, 0) and (0, 300) respectively. Town X has 100 residents, town Y has 200, and town Z has 300. A train station is to be built at coordinates (x, y), where x and y are both integers, such that the overall distance traveled by all the residents is minimized. What is (x, y)?

18. What is the cardinality of the largest subset of $\{1, 2, ..., 2022\}$ such that no integer in the subset is twice another?

19. Let C_1 be the circle of radius 1 centered at $(1, 1)$ on the xy plane. Define C_n to be the circle tangent to C_{n-1}, $x = 0$, and $y = 0$. What is the area of the shaded region?



20. For each positive integer n, define $f(n)$ to be the number of positive integers m such that $\gcd(m, n)^2 = \operatorname{lcm}(m, n)$. Compute the smallest n such that $f(n) > 10$.

21. Let $\triangle ABC$ be a triangle with $\overline{AB} = 20$ and $\overline{AC} = 22$ and circumcircle ω. Let H be the orthocenter of the triangle, and let $\overline{AA'}$ be a diameter of ω. Suppose that $\overline{CA'}$ intersects $\overline{AH}$ at T. If $BHCT$ is cyclic then the sum of all possible lengths of $\overline{BC}$ can be expressed in the form $a\sqrt{b}$ where a and b are integers and b is square-free. What is $a\sqrt{b}$?

22. Let $ABCA_1B_1C_1$ be a right regular triangular prism with triangular faces $\triangle ABC$ and $\triangle A_1B_1C_1$ and edges $\overline{AA_1}, \overline{BB_1}, \overline{CC_1}$. A sphere is tangent to sides $\overline{AB}, \overline{BC}, \overline{AC}$ at points M, N, P and to the plane that the triangle $\triangle A_1B_1C_1$ is in at point Q. Let $\measuredangle MPQ = 45^\circ$ and the distance between lines $\overline{MP}$ and $\overline{NQ}$ be equal to 1. Find the the side length of the base of the prism.

23. Evaluate

$$\sum_{j=1}^{12345} \frac{1}{24691 - 2j} \left(\prod_{k=1}^{j-1} \frac{24690 - 2k}{24691 - 2k} \right).$$

24. Consider the sequence of integers $\{a_n\}_{n \geq 1}$ constructed in the following way. $a_1 \geq 1$, and for $n \geq 2$ we have $a_n = (a_{n-1}^2 + 337a_{n-1})$ modulo 2022. We define the period of a sequence to be the smallest integer k such that there is an integer N such that for all $n \geq N$ we have $a_{n+k} = a_n$. Determine the sum of all possible periods of a_n.

25. Suppose that a, b, c are real numbers which satisfy $a^2 + b^2 + c^2 = 2022$. Let $x = \sqrt{2022 - c^2}$ and $y = \sqrt{2022 - 2ac}$. Find the minimum value of

$$\frac{xy \cdot (x + y + c)}{b^2 c}.$$