1. What is the length of the longest string of consecutive prime numbers that divide 224444220?

   Answer: 6

   Solution: We note that this is

$$2020(x^5 + x^4 + x^3 + x^2 + 1) = 2020(x + 1)(x^2 - x + 1)(x^2 + x + 1)$$

   when $x = 10$. So, we can factor it as $2020 \times 11 \times 91 \times 111 = 4 \times 5 \times 101 \times 11 \times 7 \times 13 \times 3 \times 37$. So, we count that 2,3,5,7,11,13 all divide 224444220. So, $\boxed{6}$ consecutive primes divide the number.

2. Two circles of radius r are spaced so their centers are $2r$ apart. If $A(r)$ is the area of the smallest square containing both circles, what is $\frac{A(r)}{r^2}$?

   Answer: $6 + 4\sqrt{2}$

   Solution: We can see that the two circles should be on the diagonal of the square. So, the square has diagonal of length $2r + 2r\sqrt{2}$. Thus,

$$\frac{A(r)}{r^2} = \frac{4r^2(1 + \sqrt{2})^2}{2r^2} = \boxed{6 + 4\sqrt{2}}$$

3. Define $e_0 = 1$ and $e_k = e^{e_{k-1}}$ for $k \geq 1$. Compute

$$\int_{e_4}^{e_6} \frac{1}{x(\ln x)(\ln \ln x)(\ln \ln \ln x)} dx.$$

   Answer: $e^e - 1$

   Solution: Define $\ln^0 x = x$ and $\ln^k x = \ln(\ln^{k-1} x)$ for $k \geq 1$. Notice that, by induction, the derivative of $\ln^k x$ is $\frac{1}{\ln^0 x \ln^1 x \cdots \ln^{k-1} x}$. Therefore, the above integral has antiderivative $\ln^4 x$. Evaluating at the endpoints gives $\boxed{e^e - 1}$.

4. Compute

$$\prod_{n=0}^{\infty} \sum_{k=0}^{2020} \left(\frac{1}{2020}\right)^{k(2021)^n}$$

   Answer: $\frac{2020}{2019}$

   Solution: Note that

$$\sum_{k=0}^{2020} \left(\frac{1}{2020}\right)^{k(2021)^n} = \frac{1 - (1/2020)^{2021^{n+1}}}{1 - (1/2020)^{2021^n}}$$

   Thus, our infinite product will become $\frac{1}{1 - \frac{1}{2020}} = \boxed{\frac{2020}{2019}}$

5. A potter makes a bowl, beginning with a sphere of clay and then cutting away the top and bottom of the sphere with two parallel cuts that are equidistant from the center. Finally, he hollows out the remaining shape, leaving a base at one end. Assume that the thickness of the bowl is negligible. If the potter wants the bowl to hold a volume that is $\frac{13}{27}$ of the volume of the sphere he started with, the distance from the center at which he makes his cuts should be what fraction of the radius?

Answer: $\frac{1}{3}$

Solution: Without loss of generality, we can let the radius of the sphere be 1. If the cuts are made at a distance of a from the center of the sphere, the volume of the bowl can be calculated as

$$\int_{-a}^{a} \pi(1-x^2)dx = \pi\left(x - \frac{1}{3}x^3\right)\Big|_{-a}^{a}$$
$$= \pi\left(2a - \frac{2}{3}a^3\right).$$

We want this to be equal to $\frac{13}{27} \cdot \frac{4}{3}\pi = \frac{52}{81}\pi$.

$$\pi\left(2a - \frac{2}{3}a^3\right) = \frac{52}{81}\pi$$
$$\Rightarrow 6a - 2a^3 = \frac{52}{27}$$
$$\Rightarrow a^3 - 3a + \frac{26}{27} = 0.$$

Notice that $3 \cdot \frac{1}{3} - \left(\frac{1}{3}\right)^3 = \frac{26}{27}$, so $\frac{1}{3}$ is a solution to the equation. The polynomial can be factored as $\frac{1}{27}(3a-1)(9a^2+3a-26) = 0$. The solutions to the quadratic $9a^2+3a-26$ are both not in the range $-1 \leq a \leq 1$, so the answer is $\boxed{\frac{1}{3}}$.

6. Let AB be a line segment with length $2+\sqrt{2}$. A circle ω with radius 1 is drawn such that it passes through the end point B of the line segment and its center O lies on the line segment AB. Let C be a point on circle ω such that $AC = BC$. What is the size of angle CAB in degrees?

   Answer: 22.5

   Solution: Extend AC such that it cuts the circle ω at point D and let E be the point where the line segment AB and the circle ω interesects. Notice that $\angle ADE = \angle CBA = \angle CAB$. Hence, $ED = EA = \sqrt{2}$. Since EB is a diameter of the circle, $\angle EDB = 90°$ and thus $\angle DEB = \cos^{-1}\left(\frac{ED}{EB}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = 45°$. Observe that $\angle DEB = \angle DCB = \angle CAB + \angle CBA = 2\angle CAB$. Therefore, $\angle CAB = \frac{1}{2}\angle DEB = \boxed{22.5°}$.

7. Find all possible values of $\sin x$ such that

$$4\sin(6x) = 5\sin(2x).$$

   Answer: $\mathbf{0, \pm 1, \pm\frac{\sqrt{2}}{4}, \pm\frac{\sqrt{14}}{4}}$

   Solution: Notice that $\sin 3x = 3\sin x - 4\sin^3 x$, $\cos 3x = 4\cos^3 x - 3\cos x$, and $\sin 2x = 2\sin x\cos x$. Therefore, we need to solve $(3-4\sin^2 x)(4\cos^2 x - 3) = -1$. Using the identity $\cos^2 x = 1 - \sin^2 x$ and writing $\sin x = z$, we see that we get $16z^4 - 16z^2 + \frac{z}{4}$, hence $z^2 = \frac{1}{8}, \frac{7}{8}$, which means $\sin x$ can only be $\boxed{\pm\frac{\sqrt{2}}{4}, \pm\frac{\sqrt{14}}{4}}$.

8. Frank the frog sits on the first lily pad in an infinite line of lily pads. Each lily pad besides the one first one is randomly assigned a real number from 0 to 1. Franks starting lily pad is assigned 0. Frank will jump forward to the next lily pad as long as the next pad's number is greater than his current pad's number. For example, if the first few lily pads including Frank's are numbered 0, .4, .72, .314, Frank will jump forward twice, visiting a total of 3 lily pads. What is the expected number of lily pads that Frank will visit?

   Answer: $\boldsymbol{e}$

   Solution: Let the number of lily pads Frank visits be X, note that X will only take on positive integer values, then we can write $E(X) = \sum_{i=0}^{\infty} P(X > i)$. Note that $P(X > i)$, the probability that Frank visits i or more lily pads, is the probability that the first i lily pads after Franks starting lily pad are in ascending order. Thus $P(X > i) = \frac{1}{i!}$. Then $E(X) = \sum_{i=0}^{\infty} P(X > i) = \sum_{i=0}^{\infty} \frac{1}{i!} = \boxed{e}$

9. For positive integers n and k with $k \leq n$, let

$$f(n,k) = \sum_{j=0}^{k-1} j \binom{k-1}{j} \binom{n-k+1}{k-j}.$$

   Compute the sum of the prime factors of

$$f(4,4) + f(5,4) + f(6,4) + \cdots + f(2021,4).$$

   Answer: 1881

   Solution: We first prove the following lemma.

   Lemma. *The sum*

$$\sum_{j=0}^{k-1} j \binom{k-1}{j} \binom{n-k+1}{k-j}$$

   is equal to

$$(n-1)\binom{n-2}{k-2}.$$

   Proof. Divide both quantities by $\binom{n}{k}$. Then we want to prove that

$$\frac{1}{\binom{n}{k}} \sum_{j=0}^{k-1} j \binom{k-1}{j} \binom{n-k+1}{k-j} = \frac{(n-1)\binom{n-2}{k-2}}{\binom{n}{k}}.$$

   To do this, we show that both quantities calculate the expected number of consecutive pairs of integers that would appear in a randomly selected set of k integers from $\{1, 2, \ldots, n\}$.

   Let p_j be the probability that j pairs of consecutive integers appear among the k randomly selected integers. We compute p_j through distributions: the k selected integers split the remaining $n-k$ integers into $k+1$ distinct groups. Let $a_1, a_2, \ldots, a_{k+1}$ be the number of integers in these groups. Then

$$a_1 + a_2 + \cdots + a_{k+1} = n - k.$$

If there are j pairs of consecutive integers, then exactly j of the quantities $a_2, a_3, \ldots, a_k$ must be 0. The remaining a_i with $2 \leq i \leq k$ must be positive while a_1 and a_{k+1} are allowed to be nonnegative. Letting $a_1 = a_1' - 1$ and $a_{k+1} = a_{k+1}' - 1$, we want to find the number of positive integer solutions to

$$a_1' + a_2 + \cdots + a_k + a_{k+1}' = n - k + 2,$$

where j of the variables $a_2, a_3, \ldots, a_k$ are 0. There are $\binom{k-1}{j}$ ways to choose the j variables that are equal to 0 and $\binom{n-k+2-1}{k+1-j-1} = \binom{n-k+1}{k-j}$ solutions to the resulting positive-integer equation. Thus,

$$p_j = \frac{\binom{k-1}{j}\binom{n-k+1}{k-j}}{\binom{n}{k}}.$$

It follows then that

$$\mathbb{E}[n,k] = \sum_{j=0}^{k-1} j p_j = \frac{1}{\binom{n}{k}} \sum_{j=0}^{k-1} j \binom{k-1}{j}\binom{n-k+1}{k-j}.$$

Now, we can also compute $\mathbb{E}[n,k]$ using linearity of expectation. Let X_i be the random variable that takes on the value 1 when i and $i+1$ are in the k selected integers and 0 otherwise. Then

$$\mathbb{E}[n,k] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_{n-1}].$$

The probability that i and $i+1$ both appear among the k selected integers is $\frac{\binom{n-2}{k-2}}{\binom{n}{k}}$, so

$$E[X_i] = \frac{\binom{n-2}{k-2}}{\binom{n}{k}}$$

and

$$E[n,k] = (n-1)\frac{\binom{n-2}{k-2}}{\binom{n}{k}}.$$

Hence,

$$\frac{1}{\binom{n}{k}} \sum_{j=0}^{k-1} j \binom{k-1}{j}\binom{n-k+1}{k-j} = \frac{(n-1)\binom{n-2}{k-2}}{\binom{n}{k}},$$

as desired. $\square$

The lemma gives $f(n,k) = (n-1)\binom{n-2}{k-2}$, so we have by the Hockey Stick Identity

$$\begin{aligned} f(4,4)+f(5,4)+f(6,4)+\cdots+f(2021,4) &= \sum_{n=4}^{2021}(n-1)\binom{n-2}{2} \\ &= \sum_{n=4}^{2021} 3\binom{n-1}{3} \\ &= 3\sum_{n=4}^{2021}\binom{n-1}{3} \\ &= 3\binom{2021}{4}. \end{aligned}$$

The prime factorization of $3\binom{2021}{4}$ is $3\binom{2021}{4} = 3\cdot 5\cdot 43\cdot 47\cdot 101\cdot 673\cdot 1009$

Therefore, the sum of the prime factors of $f(4,4)+f(5,4)+f(6,4)+\cdots+f(2021,4)$ is $3+5+43+47+101+673+1009 = 1881$.

10. $\triangle ABC$ has side lengths $AB = 5, AC = 10$, and $BC = 9$. The median of $\triangle ABC$ from A intersects the circumcircle of the triangle again at point D. What is $BD + CD$?

Answer: $\mathbf{\frac{135}{13}}$

Solution: Let M be the midpoint of BC and m the length of AM. In the circumcircle, we see that $\triangle BMA \sim \triangle DMC$ and $\triangle CMA \sim \triangle DMB$. This gives us the equations $\frac{\frac{9}{2}}{m} = \frac{BD}{10} \Rightarrow BD = \frac{45}{m}$ and $\frac{\frac{9}{2}}{m} = \frac{CD}{5} \Rightarrow CD = \frac{45}{2m}$. Let $BD = 2x$ and $CD = x$.

Using the Law of Cosines, we have $\cos A = \frac{(AB)^2+(AC)^2-(BC)^2}{2(AB)(AC)} = \frac{5^2+10^2-9^2}{2\times5\times10} = \frac{11}{25}$. $\angle BAC + \angle BDC = 180^\circ$, so $\cos\angle BDC = -\cos\angle BAC = -\frac{11}{25}$. Using the Law of Cosines in $\triangle BDC$, we have $BC^2 = BD^2+CD^2-2(BD)(CD)\cos\angle BDC \Rightarrow 81 = x^2(4+1+\frac{44}{25}) \Rightarrow x^2 = \frac{81\times25}{169} \Rightarrow x = \frac{45}{13}$. $BD + CD = 3x = \boxed{\frac{135}{13}}$.

11. A subset of five distinct positive integers is chosen uniformly at random from the set $\{1, 2, \ldots, 11\}$. The probability that the subset does not contain three consecutive integers can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Answer: 52

Solution: We proceed by PIE and complementary counting, finding the number of ways to select five integers from the given set such that there is at least one triplet of consecutive integers.

The five chosen integers divide the remaining six into six distinct groups. Thus, we wish to determine the number of solutions to

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 6$$

with at least one pair (a_i, a_{i+1}) equal to $(0, 0)$ for $2 \le i \le 5$.

Let A be the set of solutions in which $a_2 = a_3 = 0$, B be the set of solutions in which $a_3 = a_4 = 0$, and C be the set of solutions in which $a_4 = a_5 = 0$. Note that by setting two a_i to 0, the equation reduces to four variables summing to 6. So by stars-and-bars, $|A| = |B| = |C| = \binom{6+4-1}{4-1} = \binom{9}{3}$.

By a similar argument, we have $|A\cap B| = |B\cap C| = \binom{6+3-1}{3-1} = \binom{8}{2}$ and $|A\cap C| = \binom{6+2-1}{2-1} = \binom{7}{1}$. Finally, $|A\cap B\cap C| = \binom{6+2-1}{2-1} = \binom{7}{1}$, so by PIE,

$$|A\cup B\cup C| = 3\binom{9}{3} - 2\binom{8}{2} - \binom{7}{1} + \binom{7}{1} = 3\times 84 - 2\times 28 = 252 - 56 = 196.$$

And since there are $\binom{11}{5}$ ways to select five integers from the set $\{1, 2, \ldots, 11\}$, the probability that *no three* are consecutive is

$$1 - \frac{196}{\binom{11}{5}} = 1 - \frac{196\times 5\times 4\times 3\times 2\times 1}{11\times 10\times 9\times 8\times 7} = 1 - \frac{14}{33} = \frac{19}{33}.$$

Thus, $m + n = 19 + 33 = 52$.

12. Compute

$$\int_{-1}^{1} (x^2 + x + \sqrt{1-x^2})^2 dx.$$

Answer: $\mathbf{\frac{12}{5} + \frac{\pi}{4}}$

Solution: First, expand to get $(x^2 + x + \sqrt{1-x^2})^2 = x^4 + 1 + 2x(x^2 + \sqrt{1-x^2}) + 2x^2\sqrt{1-x^2}$. Therefore, the integral is equal to

$$\int_{-1}^{1} (x^4 + 1)dx + \int_{-1}^{1} 2x(x^2 + \sqrt{1-x^2})dx + \int_{-1}^{1} 2x^2\sqrt{1-x^2}dx.$$

Evaluating the first integral and noting that the integral of an odd function from -1 to 1 is equal to 0, the integral is equal to

$$\left[\frac{x^5}{5} + x\right]_{-1}^{1} + \int_{-1}^{1} 2x^2\sqrt{1-x^2}dx = \frac{12}{5} + \int_{-1}^{1} 2x^2\sqrt{1-x^2}dx.$$

To solve for the remaining integral, substituting $x = sin(u)$ and applying trigonometric identities yields

$$\int_{-1}^{1} 2x^2\sqrt{1-x^2}dx = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^2(u)cos^2(u)du = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} sin^2(2u)du = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 - cos(4u)}{2}du.$$

Solving, this is equal to

$$\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1 - cos(4u)du = \left[\frac{u}{4} - \frac{sin(4u)}{16}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

Therefore, the value of the integral is $\frac{12}{5} + \frac{\pi}{4}$.

13. Three friends, Xander, Yulia, and Zoe, have each planned to visit the same cafe one day. If each person arrives at the cafe at a random time between 2 PM and 3 PM and stays for 15 minutes, what is the probability that all three friends will be there at the same time at some point?

Answer: $\frac{5}{32}$

Solution: We can use a cube to represent all possible combinations of arrival times, with one axis for each friend. To simplify calculations, let the side length of the cube be 4 units (1 unit = 15 minutes). On the axes, 0 represents arriving at 2 PM and 4 represents arriving at 3 PM. Let us first consider the probability that Xander and Yulia will be at the cafe at the same time. This situation can be represented inside a 4×4 square, with the probability being the area of the hexagon bounded by $|x - y| \leq 1$ and the boundaries of the square.

Suppose that the arrival times for Xander and Yulia have been determined as x and y. Zoe must arrive at the earliest no more than 15 minutes (1 unit) before the later of these two, and no later than 15 minutes (1 unit) after the earlier of these two. So, if $x < y$, Zoe must arrive between $y - 1$ and $x + 1$. If $x > y$, Zoe must arrive between $x - 1$ and $y + 1$.

Following these restrictions, we can draw cross-sections parallel to the yz-plane for given values of x. Suppose that Xander arrives at x (in units). Then, $x - 1 \leq y \leq x + 1$. First consider the case in which $y < x$. We must have $x - 1 \leq z \leq y + 1$. As y varies from $x - 1$ to x, the lower bound for z remains constant at $x - 1$ while the upper bound, $y + 1$, increases from x to $x + 1$. Next consider the case in which $y > x$. We must have $y - 1 \leq z \leq x + 1$. As y varies from x to $x + 1$, the lower bound, $y - 1$, for z increases from $x - 1$ to x while the upper bound remains constant at $x + 1$. If we sketch the resulting area as y varies from $x - 1$ to $x + 1$, we have a hexagon of area 3 (similar in shape to the hexagon in two dimensions mentioned above).

However, the cross-section is not always a hexagon because it gets truncated at the extremes of the possible arrival times. This is because the friends are not allowed to arrive before 2 PM or after 3 PM. Let us consider the extremes. If $x = 0$, then y and z must both be between 0 and 1. The cross-section is thus a unit square cut from rest of the hexagon. As x increases to 1, the hexagon "emerges" from the corner of the plane until it is completely included in the cross-section once $x \geq 1$. Then, for $x \geq 3$, the hexagon becomes truncated in the same way until the cross-section again is a unit square at $x = 4$.

With these cross-sections put together, the resulting polyhedron represents the possible arrival times that satisfy the condition. The middle section of the polyhedron, for $1 \leq x \leq 3$, is an oblique prism. The bases are the hexagonal cross-section with area 3 and the height is 2, so the volume is 6. The two ends left over are identical polyhedrons of height 1, with one base being a unit square and the other base the hexagonal cross-section. We can calculate the volume of this polyhedron by cutting it into three pieces: a triangular prism with base area $\frac{1}{2}$, half of a rectangular prism with base area 2, and an oblique triangular prism with base area $\frac{1}{2}$. Adding up the volumes, we get $\frac{1}{2} + \frac{1}{2} \times 2 + \frac{1}{2} = 2$. Finally, the volume of the entire polyhedron is $2 \times 2 + 6 = 10$.

The total volume of the cube is $4^3 = 64$, so the probability we seek is $\frac{10}{64} = \boxed{\frac{5}{32}}$.

14. Jim the Carpenter starts with a wooden rod of length 1 unit. Jim will cut the middle $\frac{1}{3}$ of the rod and remove it, creating 2 smaller rods of length $\frac{1}{3}$. He repeats this process, randomly

choosing a rod to split into 2 smaller rods. Thus, after two such splits, Jim will have 3 rods of length $\frac{1}{3}$, $\frac{1}{9}$, and $\frac{1}{9}$. After 3 splits, Jim will either have 4 rods of lengths $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}$ or $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{27}$, What is the expected value of the total length of the rods after 5 splits?

Answer: $\mathbf{\frac{308}{729}}$

Solution: Let $E(i)$ denote the expected value of the total length of the rods after i splits. Then $E(0) = 1, E(1) = \frac{2}{3}, E(2) = \frac{5}{9}$.... Let $D(i)$ denote the expected value of the length removed by the ith split. Thus, $D(1) = \frac{1}{3}, D(2) = \frac{1}{9}$.... Then, we also have that $E(i) = E(i-1) - D(i)$. Note that $D(i) = \frac{E(i-1)}{3i}$ as the expected length removed is 1/3 the expected length of a single rod in the previous stage. Then, $D(3) = \frac{E(2)}{3*3} = \frac{5}{81}.E(3) = E(2) - D(3) = \frac{40}{81}.D(4) = \frac{E(3)}{3*4} = \frac{10}{243}.E(4) = E(3) - D(4) = \frac{110}{243}.D(5) = \frac{E(4)}{3*5} = \frac{22}{729}$. Lastly, $E(5) = E(4) - D(5) = \boxed{\frac{308}{729}}$

15. Robin is at an archery range. There are 10 targets, each of varying difficulty. If Robin spends t seconds concentrating on target n where $1 \leq n \leq 10$, he has a probability $p = 1 - e^{-t/n}$ of hitting the target. Hitting target n gives him n points and he is alloted a total of 60 seconds. If he has at most one attempt on each target, and he allots time to concentrate on each target optimally to maximize his point total, what is the expected value of the number of points Robin will get? (Assume no time is wasted between shots).

Answer: $\mathbf{55 - 55e^{-12/11}}$

Solution: Assume that Robin allots time t_k to concentrate on target k. Then the expected value of his score is:

$$\sum_{k=1}^{10} k(1 - e^{t_k/k}) = 55 - (e^{-t_1} + 2e^{-t_2/2} + \ldots + 10e^{-t_{10}/10})$$

By AM-GM, we have

$$e^{-t_1} + 2e^{-t_2/2} + \ldots + 10e^{-t_{10}/10} \geq 55\sqrt[55]{e^{-t_1-\ldots-t_{10}}} = 55\sqrt[55]{e^{-60}}$$

Therefore, the maximum expected value of his score is $55 - 55e^{\frac{-12}{11}}$. This is achieved when he allots $12k/11$ seconds to concentrating on target k.