Time limit: 50 minutes.
Instructions: For this test, you work in teams of eight to solve 15 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.
No calculators.

1. What is the length of the longest string of consecutive prime numbers that divide 224444220?

2. Two circles of radius r are spaced so their centers are $2r$ apart. If $A(r)$ is the area of the smallest square containing both circles, what is $\frac{A(r)}{r^2}$?

3. Define $e_0 = 1$ and $e_k = e^{e_{k-1}}$ for $k \geq 1$. Compute
$$\int_{e_4}^{e_6} \frac{1}{x(\ln x)(\ln \ln x)(\ln \ln \ln x)} dx.$$

4. Compute
$$\prod_{n=0}^{\infty} \sum_{k=0}^{2020} \left(\frac{1}{2020}\right)^{k(2021)^n}$$

5. A potter makes a bowl, beginning with a sphere of clay and then cutting away the top and bottom of the sphere with two parallel cuts that are equidistant from the center. Finally, he hollows out the remaining shape, leaving a base at one end. Assume that the thickness of the bowl is negligible. If the potter wants the bowl to hold a volume that is $\frac{13}{27}$ of the volume of the sphere he started with, the distance from the center at which he makes his cuts should be what fraction of the radius?

6. Let AB be a line segment with length $2 + \sqrt{2}$. A circle ω with radius 1 is drawn such that it passes through the end point B of the line segment and its center O lies on the line segment AB. Let C be a point on circle ω such that $AC = BC$. What is the size of angle CAB in degrees?

7. Find all possible values of $\sin x$ such that
$$4\sin(6x) = 5\sin(2x).$$

8. Frank the frog sits on the first lily pad in an infinite line of lily pads. Each lily pad besides the one first one is randomly assigned a real number from 0 to 1. Franks starting lily pad is assigned 0. Frank will jump forward to the next lily pad as long as the next pad's number is greater than his current pad's number. For example, if the first few lily pads including Frank's are numbered 0, .4, .72, .314, Frank will jump forward twice, visiting a total of 3 lily pads. What is the expected number of lily pads that Frank will visit?

9. For positive integers n and k with $k \leq n$, let
$$f(n,k) = \sum_{j=0}^{k-1} j \binom{k-1}{j} \binom{n-k+1}{k-j}.$$

Compute the sum of the prime factors of

$$f(4,4)+f(5,4)+f(6,4)+\cdots+f(2021,4).$$

10. $\triangle ABC$ has side lengths $AB=5, AC=10$, and $BC=9$. The median of $\triangle ABC$ from A intersects the circumcircle of the triangle again at point D. What is $BD+CD$?

11. A subset of five distinct positive integers is chosen uniformly at random from the set $\{1,2,\ldots,11\}$. The probability that the subset does not contain three consecutive integers can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

12. Compute

$$\int_{-1}^{1}(x^2+x+\sqrt{1-x^2})^2dx.$$

13. Three friends, Xander, Yulia, and Zoe, have each planned to visit the same cafe one day. If each person arrives at the cafe at a random time between 2 PM and 3 PM and stays for 15 minutes, what is the probability that all three friends will be there at the same time at some point?

14. Jim the Carpenter starts with a wooden rod of length 1 unit. Jim will cut the middle $\frac{1}{3}$ of the rod and remove it, creating 2 smaller rods of length $\frac{1}{3}$. He repeats this process, randomly choosing a rod to split into 2 smaller rods. Thus, after two such splits, Jim will have 3 rods of length $\frac{1}{3}$, $\frac{1}{9}$, and $\frac{1}{9}$. After 3 splits, Jim will either have 4 rods of lengths $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}$, $\frac{1}{9}$ or $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, $\frac{1}{27}$, What is the expected value of the total length of the rods after 5 splits?

15. Robin is at an archery range. There are 10 targets, each of varying difficulty. If Robin spends t seconds concentrating on target n where $1\le n\le 10$, he has a probability $p=1-e^{-t/n}$ of hitting the target. Hitting target n gives him n points and he is alloted a total of 60 seconds. If he has at most one attempt on each target, and he allots time to concentrate on each target optimally to maximize his point total, what is the expected value of the number of points Robin will get? (Assume no time is wasted between shots).