**Time limit:** 50 minutes.
**Instructions:** This test contains 10 short answer questions. All answers must be expressed in simplest form unless specified otherwise. Only answers written inside the boxes on the answer sheet will be considered for grading.
**No calculators.**

1. A rectangular pool has diagonal 17 *units* and area 120 $units^2$. Joey and Rachel start on opposite sides of the pool when Rachel starts chasing Joey. If Rachel runs 5 units/sec faster than Joey, how long does it take for her to catch him?

2. Alice plays a game with her standard deck of 52 cards. She gives all of the cards number values where Aces are 1's, royal cards are 10's and all other cards are assigned their face value. Every turn she flips over the top card from her deck and creates a new pile. If the flipped card has value $v$, she places $12 - v$ cards on top of the flipped card. For example: if she flips the 3 of diamonds then she places 9 cards on top. Alice continues creating piles until she can no longer create a new pile. If the number of leftover cards is 4 and there are 5 piles, what is the sum of the flipped over cards?

3. There are 5 people standing at $(0, 0)$, $(3, 0)$, $(0, 3)$, $(-3, 0)$, and $(-3, 0)$ on a coordinate grid at a time $t = 0$ seconds. Each second, every person on the grid moves exactly 1 unit up, down, left, or right. The person at the origin is infected with covid-19, and if someone who is not infected is at the same lattice point as a person who is infected, at any point in time, they will be infected from that point in time onwards. (Note that this means that if two people run into each other at a non-lattice point, such as $(0, 1.5)$, they will not infect each other.) What is the maximum possible number of infected people after $t = 7$ seconds?

4. Kara gives Kaylie a ring with a circular diamond inscribed in a gold hexagon. The diameter of the diamond is 2mm. If diamonds cost $\$100/mm^2$ and gold costs $\$50/mm^2$, what is the cost of the ring?

5. Find the number of three-digit integers that contain at least one 0 or 5. The leading digit of the three-digit integer cannot be zero.

6. What is the sum of the solutions to $\frac{x+8}{5x+7} = \frac{x+8}{7x+5}$

7. Let $BC$ be a diameter of a circle with center $O$ and radius 4. Point $A$ is on the circle such that $AOB = 45°$. Point $D$ is on the circle such that line segment $OD$ intersects line segment $AC$ at $E$ and $OD$ bisects $\angle AOC$. Compute the area of $ADE$, which is enclosed by line segments $AE$ and $ED$ and minor arc $\widehat{AD}$.

8. William is a bacteria farmer. He would like to give his fiancé 2021 bacteria as a wedding gift. Since he is an intelligent and frugal bacteria farmer, he would like to add the least amount of bacteria on his favourite infinite plane petri dish to produce those 2021 bacteria.

   The infinite plane petri dish starts off empty and William can add as many bacteria as he wants each day. Each night, all the bacteria reproduce through binary fission, splitting into two. If he has infinite amount of time before his wedding day, how many bacteria should he add to the dish in total to use the least number of bacteria to accomplish his nuptial goals?

9. The frozen yogurt machine outputs yogurt at a rate of 5 $\text{froyo}^3$/second. If the bowl is described by $z = x^2 + y^2$ and has height 5 froyos, how long does it take to fill the bowl with frozen yogurt?

10. Prankster Pete and Good Neighbor George visit a street of 2021 houses (each with individual mailboxes) on alternate nights, such that Prankster Pete visits on night 1 and Good Neighbor George visits on night 2, and so on. On each night $n$ that Prankster Pete visits, he drops a packet of glitter in the mailbox of every $n^{\text{th}}$ house. On each night $m$ that Good Neighbor George visits, he checks the mailbox of every $m^{\text{th}}$ house, and if there is a packet of glitter there, he takes it home and uses it to complete his art project. After the $2021^{\text{th}}$ night, Prankster Pete becomes enraged that none of the houses have yet checked their mail. He then picks three mailboxes at random and takes out a single packet of glitter to dump on George's head, but notices that all of the mailboxes he visited had an odd number of glitter packets before he took one. In how many ways could he have picked these three glitter packets? Assume that each of these three was from a different house, and that he can only visit houses in increasing numerical order.

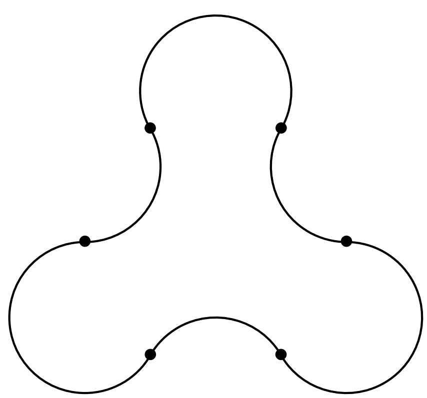
11. The taxi-cab length of a line segment with endpoints $(x_1, y_1)$ and $(x_2, y_2)$ is $|x_1 - x_2| + |y_1 - y_2|$. Given a series of straight line segments connected head-to-tail, the taxi-cab length of this path is the sum of the taxi-cab lengths of its line segments. A goat is on a rope of taxi-cab length $\frac{7}{2}$ tied to the origin, and it can't enter the house, which is the three unit squares enclosed by $(-2, 0), (0, 0), (0, -2), (-1, -2), (-1, -1), (-2, -1)$. What is the area of the region the goat can reach? (Note: the rope can't "curve smoothly"—it must bend into several straight line segments.)

12. Parabola $P$, $y = ax^2 + c$ has $a > 0$ and $c < 0$. Circle $C$, which is centered at the origin and lies tangent to $P$ at $P$'s vertex, intersects $P$ at only the vertex. What is the maximum value of $a$, possibly in terms of $c$?

13. Emma has the five letters: A, B, C, D, E. How many ways can she rearrange the letters into words? Note that the order of words matter, ie ABC DE and DE ABC are different.

14. Seven students are doing a holiday gift exchange. Each student writes their name on a slip of paper and places it into a hat. Then, each student draws a name from the hat to determine who they will buy a gift for. What is the probability that no student draws himself/herself?

15. We model a fidget spinner as shown below (include diagram) with a series of arcs on circles of radii 1. What is the area swept out by the fidget spinner as it's turned 60°?



16. Let a,b,c be the sides of a triangle such that

$$gcd(a, b) = 3528, gcd(b, c) = 1008, gcd(a, c) = 504$$

Find the value of $a * b * c$. Write your answer as a prime factorization.

17. Let the roots of the polynomial $f(x) = 3x^3 + 2x^2 + x + 8 = 0$ be $p$, $q$, and $r$. What is the sum $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$?

18. Two students are playing a game. They take a deck of five cards numbered 1 through 5, shuffle them, and then place them in a stack facedown, turning over the top card next to the stack. They then take turns either drawing the card at the top of the stack into their hand, showing the drawn card to the other player, or drawing the card that is faceup, replacing it with the card on the top of the pile. This is repeated until all cards are drawn, and the player with the largest sum for their cards wins. What is the probability that the player who goes second wins, assuming optimal play?

19. Compute the sum of all primes $p$ such that $2^p + p^2$ is also prime.

20. In how many ways can one color the 8 vertices of an octagon each red, black, and white, such that no two adjacent sides are the same color?

21. If $f = \cos(\sin(x))$. Calculate the sum

$$\sum_{n=0}^{2021} f''(n\pi).$$

22. Find all real values of A that minimize the difference between the local maximum and local minimum of $f(x) = (3x^2 - 4)(x - A + \frac{1}{A})$.

23. Bessie is playing a game. She labels a square with vertices labeled A, B, C, D in clockwise order. There are 7 possible moves: she can rotate her square 90 degrees about the center, 180 degrees about the center, 270 degrees about the center; or she can flip across diagonal AC, flip across diagonal BD, flip the square horizontally (flip the square so that vertices A and B are switched and vertices C and D are switched), or flip the square vertically (vertices B and C are switched, vertices A and D are switched). In how many ways can Bessie arrive back at the original square for the first time in 3 moves?

24. A positive integer is called *happy* if the sum of its digits equals the two-digit integer formed by its two leftmost digits. Find the number of 5-digit happy integers.

25. Compute:

$$\frac{\sum_{i=0}^{\infty} \frac{(2\pi)^{4i+1}}{(4i+1)!}}{\sum_{i=0}^{\infty} \frac{(2\pi)^{4i+1}}{(4i+3)!}}$$

26. Suppose points $A, B, C, D$ lie on a circle $\omega$ with radius 4 such that $ABCD$ is a quadrilateral with $AB = 6, AC = 8, AD = 7$. Let $E$ and $F$ be points on $\omega$ such that $AE$ and $AF$ are respectively the angle bisectors of $\angle BAC$ and $\angle DAC$. Compute the area of quadrilateral $AECF$.

27. Let $P(x) = x^2 - ax + 8$ with $a$ a positive integer, and suppose that $P$ has two distinct real roots $r$ and $s$. Points $(r, 0)$, $(0, s)$, and $(t, t)$ for some positive integer $t$ are selected on the coordinate plane to form a triangle with an area of 2021. Determine the minimum possible value of $a + t$.

28. A quartic $p(x)$ has a double root at $x = -\frac{21}{4}$, and $p(x) - 1344x$ has two double roots each $\frac{1}{4}$ less than an integer. What are these two double roots?

29. Consider pentagon ABCDE. How many paths are there from vertex A to vertex E where no edge is repeated and does not go through E.

30. Let $a_1, a_2, \ldots$ be a sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n = 4$. Compute the maximum possible value of $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{2^n}$ (assume this always converges).

31. Define function $f(x) = x^4 + 4$. Let

$$P = \prod_{k=1}^{2021} \frac{f(4k-1)}{f(4k-3)}.$$

Find the remainder when $P$ is divided by 1000.

32. Reduce the following expression to a simplified rational:

$$\cos^7 \frac{\pi}{9} + \cos^7 \frac{5\pi}{9} + \cos^7 \frac{7\pi}{9}$$

33. Lines $\ell_1$ and $\ell_2$ have slopes $m_1$ and $m_2$ such that $0 < m_2 < m_1$. $\ell_1'$ and $\ell_2'$ are the reflections of $\ell_1$ and $\ell_2$ about the line $\ell_3$ defined by $y = x$. Let $A = \ell_1 \cap \ell_2 = (5, 4)$, $B = \ell_1 \cap \ell_3$, $C = \ell_1' \cap \ell_2'$ and $D = \ell_2 \cap \ell_3$. If $\frac{4-5m_1}{-5-4m_1} = m_2$ and $\frac{(1+m_1^2)(1+m_2^2)}{(1-m_1)^2(1-m_2)^2} = 41$, compute the area of quadrilateral $ABCD$.

34. Suppose $S(m,n) = \sum_{i=1}^{m} (-1)^i i^n$. Compute the remainder when $S(2020, 4)$ is divided by $S(1010, 2)$.

35. Let N be the number of ways to place the numbers $1, 2, ..., 12$ on a circle such that every pair of adjacent numbers has greatest common divisor 1. What is N/144? (Arrangements that can be rotated to yield each other are the same).

36. Compute the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\binom{2n}{2}} = \frac{1}{\binom{2}{2}} - \frac{1}{\binom{4}{2}} + \frac{1}{\binom{6}{2}} - \frac{1}{\binom{8}{2}} - \frac{1}{\binom{10}{2}} + \frac{1}{\binom{12}{2}} + \cdots.$$