Time limit: 50 minutes.
Instructions: For this test, you work in teams of eight to solve 15 short answer questions.
No calculators.

1. Find the sum of the largest and smallest value of the following function: $f(x) = |x - 23| + |32 - x| - |12 + x|$ where the function has domain [-37,170].

2. Given a convex equiangular hexagon with consecutive side lengths of $9, a, 10, 5, 5, b$, where a and b are whole numbers, find the area of the hexagon.

3. Let $S = \{1, 2, \ldots, 100\}$. Compute the minimum possible integer n such that, for any subset $T \subseteq S$ with size n, every integer a in S satisfies the relation $a \equiv bc \bmod 101$, for some choice of integers b, c in T.

4. Let C be the circle of radius 2 centered at $(4, 4)$ and let L be the line $x = -2$. The set of points equidistant from C and from L can be written as $ax^2 + by^2 + cxy + dx + ey + f = 0$ where a, b, c, d, e, f are integers and have no factors in common. What is $|a + b + c + d + e + f|$?

5. If a is picked randomly in the range $\left(\frac{1}{4}, \frac{3}{4}\right)$ and b is chosen such that

$$\int_a^b \frac{1}{x^2} dx = 1,$$

compute the expected value of $b - a$.

6. Let $A_1, A_2, \ldots, A_{2020}$ be a regular 2020-gon with a circumcircle C of diameter 1. Now let P be the midpoint of the small-arc $A_1 - A_2$ on the circumcircle C. Then find:

$$\sum_{i=1}^{2020} |PA_i|^2$$

7. A certain party of 2020 people has the property that, for any 4 people in the party, there is at least one person of those 4 that is friends with the other three (assume friendship is mutual). Call a person in the party a *politician* if they are friends with the other 2019 people in the party. If n is the number of politicians in the party, compute the sum of the possible values of n.

8. Let S be an n-dimensional hypercube of sidelength 1. At each vertex draw a hypersphere of radius $\frac{1}{2}$; let Ω be the set of these hyperspheres. Consider a hypersphere Γ centered at the center of the cube that is externally tangent to all the hyperspheres in Ω. For what value of n does the volume of Γ equal to the sum of the volumes of the hyperspheres in Ω.

9. Solve for C:

$$\frac{2\pi}{3} = \int_0^1 \frac{1}{\sqrt{Cx - x^2}} dx.$$

10. Nathan and Konwoo are both standing in a plane. They each start at (0,0). They play many games of rock-paper-scissors. After each game, the winner will move one unit up, down, left, or right, chosen randomly. The outcomes of each game are independent, however, Konwoo is twice as likely to win a game as Nathan. After 6 games, what is the probability that Konwoo is located at the same point as Nathan? (For example, they could have each won 3 games and both be at (1,2).)

11. Suppose that x, y, z are real positive numbers such that $\left(1 + x^4y^4\right)e^z + \left(1 + 81e^{4z}\right)x^4e^{-3z} = 12x^3y$. Find all possible values of $x + y + z$.

12. Given a large circle with center (x_0, y_0), one can place three smaller congruent circles with centers $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ that are pairwise externally tangent to each other and all internally tangent to the outer circle. If this placement makes $x_0 = x_1$ and $y_1 > y_0$, we call this an "up-split". Otherwise, if the placement makes $x_0 = x_1$ and $y_1 < y_0$, we call it a "down-split."

    Alice starts at the center of a circle C with radius 1. Alice first walks to the center of the upper small circle C_u of an up-slitting of C. Then, Alice turns right to walk to the center of the upper-right small circle of a down-splitting of C_d. Alice continues this process of turning right and walking to the center of a new circle created by alternatingly up- and down-splitting. Alice's path will form a spiral converging to (x_A, y_A). On the otherhand, Bob always up-splits the circle he is in the center of, turns right and finds the center of the next small circle. His path will converges to (x_B, y_B). Compute $\left|\frac{1}{x_A} - \frac{1}{x_B}\right|$.

13. Compute the sum of all natural numbers b less than 100 such that b is divisible by the number of factors of the base-10 representation of 2020_b.

14. Iris is playing with her random number generator. The number generator outputs real numbers from 0 to 1. After each output, Iris computes the sum of her outputs, if that sum is larger than 2, she stops. What is the expected number of outputs Iris will receive before she stops?

15. Evaluate

$$\int_0^{\frac{\pi}{2}} \ln\left(9\sin^2\theta + 121\cos^2\theta\right) d\theta$$