1. A circle with radius 1 is circumscribed by a rhombus. What is the minimum possible area of this rhombus?

   Answer: 4

   Solution: Work with coordinate axes. Define the unit circle to be $x^2 + y^2 = 1$. An arbitrary rhombus circumscribing a circle with radius 1 will be tangent at four total points, two at $x_0 > 0$ and two at $-x_0$. Then $A = 2\csc\theta\sec\theta = 4\frac{1}{2\sin\theta} \leq 4$, so the minimum value of A is 4.

2. Let $\triangle ABC$ be a right triangle with $\angle ABC = 90°$. Let the circle with diameter BC intersect AC at D. Let the tangent to this circle at D intersect AB at E. What is the value of $\frac{AE}{BE}$?

   Answer: 1

   Solution: Let O be the center of the circle with diameter BC. Then $OC = OD$, so $\triangle COD$ is isosceles with $\angle ODC = \angle OCD$. Since $OB \perp AB$, AB is tangent to the circle so $\angle EBD = \angle OCD$. Also, ED is a tangent so $\angle EDO = 90°$. But $\angle EBO = 90°$, so $EDOB$ is cyclic. It follows that $\angle EOD = \angle EBD = \angle OCD = \angle ODC$. This implies that $OE||AC$. Since O is the midpoint of BC, E must be the midpoint of AB. Therefore, $\dfrac{AE}{BE} = \boxed{1}$.

3. Square $ABCD$ has side length 4. Points P and Q are located on sides BC and CD, respectively, such that $BP = DQ = 1$. Let AQ intersect DP at point X. Compute the area of triangle PQX.

   Answer: $\frac{45}{38}$

   Solution: Notice that the desired area is $[PQD] - [QDX]$. By the standard area of a triangle formula, $[PQD] = \frac{1}{2} \cdot 1 \cdot 3 = \frac{3}{2}$. Let $\angle QDX = \angle CDP = \theta$. Since triangle PCD is a $3-4-5$ right triangle, we have $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$. Now by the sine area formula, $[QDA] = 2 = [QDX] + [XDA] = \frac{1}{2} \cdot DX \cdot (\sin\theta + 4\cos\theta)$, so solving for DX gives $DX = \frac{20}{19}$. Thus $[QDX] = \frac{1}{2} \cdot 1 \cdot \frac{20}{19} \cdot \frac{3}{5} = \frac{6}{19}$. Our answer is $\frac{3}{2} - \frac{6}{19} = \boxed{\dfrac{45}{38}}$.

4. Let $ABCD$ be a quadrilateral such that $AB = BC = 13$, $CD = DA = 15$ and $AC = 24$. Let the midpoint of AC be E. What is the area of the quadrilateral formed by connecting the incenters of ABE, BCE, CDE, and DAE?

   Answer: 25

   Solution: Since E is the midpoint of AC, $AE = CE = 12$. Also, from $AB = BC$ and $CD = DA$, we see that $ABCD$ is a kite and $AC \perp BD$. By the Pythagorean Theorem on the four right triangles, we find that $AE = 5$ and $DE = 9$.

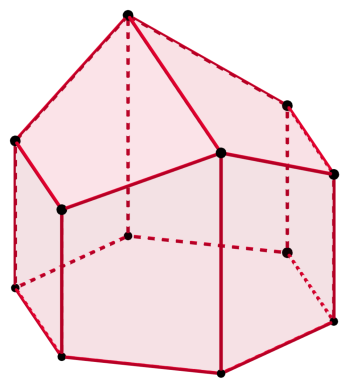
   Let W, X, Y, and Z be the incenters of $\triangle ABE$, $\triangle BCE$, $\triangle CDE$ and $\triangle DAE$ respectively. Note that $WX||AC$ and $YZ||AC$ and by symmetry, $WZ = XY$, so $WXYZ$ is an isosceles trapezoid. The semiperimeter of $\triangle ABE$ is $\frac{5+12+13}{2} = 15$, so the inradius is $15 - 13 = 2$. Similarly, we can compute that the inradius of $\triangle CDE$ is $\frac{9+12+15}{2} - 15 = 3$. It follows that $WX = 2(2) = 4$ and $YZ = 2(3) = 6$. Drawing perpendiculars from W to AC and Z to AC, we see that these perpdiculars are exactly the inradii of $\triangle ABE$ and $\triangle DAE$ respectively, so the height of the trapezoid is $2 + 3 = 5$. Thus, the area of $WXYZ$ is $\frac{10(5)}{2} = \boxed{25}$.

5. Find the smallest possible number of edges in a convex polyhedron that has an odd number of edges in total has an even number of edges on each face.

   Answer: 19

Solution: Because each edge is part of two distinct faces, we can think of a face with $2k$ edges as contributing k edges to the total edge count of the polyhedron. Then in order for the total edge count to be odd, we see that there must be an odd number of faces that have 2 (mod 4) edges.

Since our goal is to minimize the number of edges, note that the smallest possible 2 (mod 4) face is a hexagon. Let us attempt to construct an example starting with a single hexagon as the only 2 (mod 4) face. This necessitates having at least six other faces – one for each edge of the hexagon. Since all must be 0 (mod 4) faces, our best bet is to make them all quadrilaterals. In order to minimize the number of loose edges, we make every two quadrilaterals that are adjacent along the hexagon share an edge. We now have six loose edges left to cover. Note that we can do this with two more quadrilaterals. The resulting polyhedron has 19 edges in total, with one hexagonal face and eight quadrilateral faces:



We now claim this is the minimum possible number of edges for such a polyhedron. Indeed, the construction argument already explains why it is the optimal solution with exactly one hexagon and no other 2 (mod 4) faces. Furthermore, notice that the presence of an octagon or a decagon necessitates at least 8 other faces and thus at least 16 other edges, which already surpasses the 19 edges in our example. So we restrict our search to polyhedrons of hexagons and quadrilaterals. In particular, the only case we have left to rule out is three or more hexagons – but introducing three hexagons already forces at least 15 edges. One can quickly convince oneself that strictly more than 4 additional edges are needed to close a polyhedron. (To rigorize this, we can consider three cases: if there exists a hexagon not adjacent to the other hexagons, if the three hexagons are pairwise adjacent but don't share a vertex, or if all three hexagons share a vertex.) We conclude that the answer is 19.

6. Consider triangle ABC on the coordinate plane with $A = (2, 3)$ and $C = (\frac{96}{13}, \frac{207}{13})$. Let B be the point with the smallest possible y-coordinate such that $AB = 13$ and $BC = 15$. Compute the coordinates of the incenter of triangle ABC.

Answer: $\mathbf{(8, 7)}$

Solution: First, note that

$$AC = \sqrt{\left(\frac{96}{13} - 2\right)^2 + \left(\frac{207}{13} - 3\right)^2} = \sqrt{\left(\frac{70}{13}\right)^2 + \left(\frac{168}{13}\right)^2} = \sqrt{\left(\frac{14}{13}\right)^2 (5^2 + 12^2)} = 14$$

so ABC is a 13-14-15 triangle. Using Heron's Formula, we have that the area of ABC is 84. Then, if r is the inradius, $\frac{13+14+15}{2} \cdot r = 84 \implies r = 4$. Furthermore, we can draw a perpendicular from B to AC to split the 13-14-15 triangle into a 5-12-13 triangle and a 9-12-15 triangle. It follows that $\tan BAC = \frac{12}{5}$. But the slope of line AC is

$$\frac{\frac{168}{13}}{\frac{70}{13}} = \frac{168}{70} = \frac{12}{5}$$

so in fact side AB is parallel to the x-axis.

Let I be the incenter and let X be the point of tangency of the incircle to AB. We have that $AX = \frac{13+14+15}{2} - 15 = 6$, so $X = (8,3)$. But $IX \perp AB$ and $IX = 4$, so $X = \boxed{(8,7)}$.

7. Let ABC be an acute triangle with $BC = 4$ and $AC = 5$. Let D be the midpoint of BC, E be the foot of the altitude from B to AC, and F be the intersection of the angle bisector of $\angle BCA$ with segment AB. Given that AD, BE, and CF meet at a single point P, compute the area of triangle ABC. Express your answer as a common fraction in simplest radical form.

Answer: $\mathbf{20\sqrt{14}/9}$

Solution: By the Angle Bisector Theorem, $\frac{BF}{AF} = \frac{4}{5}$. Then by Ceva's theorem we see that $\frac{CE}{AE} = \frac{4}{5}$, so $CE = \frac{20}{9}$ and $AE = \frac{25}{9}$. By the Pythagorean theorem, $BE = \frac{8\sqrt{14}}{9}$, so the area of $\triangle ABC$ is $\frac{1}{2} \cdot 5 \cdot \frac{8\sqrt{14}}{9} = \frac{20\sqrt{14}}{9}$.

8. Consider an acute angled triangle $\triangle ABC$ with side lengths 7, 8, and 9. Let H be the orthocenter of ABC. Let Γ_A, Γ_B, and Γ_C be the circumcircles of $\triangle BCH$, $\triangle CAH$, and $\triangle ABH$ respectively. Find the area of the region $\Gamma_A \cup \Gamma_B \cup \Gamma_C$ (the set of all points contained in at least one of Γ_A, Γ_B, and Γ_C).

Answer: $\mathbf{\frac{441\pi}{10} + 24\sqrt{5}}$

Solution: Let H_A be the reflection of H across side BC. Note that $\angle AH_AB = \angle BHH_A = 90° - \angle HBC = \angle ACB$, so H_A lies on Γ, the circumcircle of ABC. In other words, the circumcircle of BH_AC is precisely Γ. So Γ_A – the circumcircle of BHC – is the reflection of Γ across side BC. Similarly, Γ_B and Γ_C are the reflections of Γ across sides CA and AB.

Let O_A, O_B, and O_C be the centers of Γ_A, Γ_B, and Γ_C. We can write the area of $\Gamma_A \cup \Gamma_B \cup \Gamma_C$ as the area of hexagon $AO_CBO_ACO_B$ plus the three external circular sectors AO_CB, BO_AC, and CO_AB. Notice that $\angle AO_CB + \angle BO_AC + \angle CO_AB = \angle AOB + \angle BOC + \angle COA = 360°$, so the sum of the areas of these three sectors is precisely twice the area of Γ. Furthermore, note that $[AO_CBO_ACO_B] = [AO_CB] + [BO_AC] + [CO_BA] + [ABC] = [AOB] + [BOC] + [COA] + [ABC] = 2[ABC]$.

We can compute $[ABC] = 12\sqrt{5}$ by Heron's, and then the circumradius R is $\frac{7\cdot 8\cdot 9}{4[ABC]} = \frac{21\sqrt{5}}{10}$. Thus the area of Γ is $\pi R^2 = \frac{441\pi}{20}$. So our final answer is $2\pi R^2 + 2[ABC] = \frac{441\pi}{10} + 24\sqrt{5}$.

9. Let ABC be a right triangle with hypotenuse AC. Let G be the centroid of this triangle and suppose that we have $AG^2 + BG^2 + CG^2 = 156$. Find AC^2.

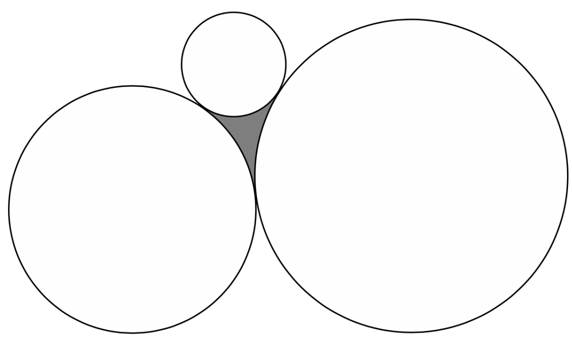
Answer: 234

Solution: Let $AB = x$ and $BC = y$. Let D, E, F be the midpoints of BC, AC, AB respectively. Since G is a centroid, we have $AG = 2GD, BG = 2GE, CG = 2GF$. Also, since ABC is a right triangle and E is the midpoint of AC, we must have $BE = EA = EC = \frac{1}{2}AC$. Hence,

$$\begin{aligned} AG^2 + BG^2 + CG^2 &= \left(\frac{2}{3}AD\right)^2 + \left(\frac{2}{3}BE\right)^2 + \left(\frac{2}{3}CF\right)^2 \\ &= \frac{4}{9}\left(x^2 + \left(\frac{y}{2}\right)^2 + \frac{x^2+y^2}{4} + y^2 + \left(\frac{x}{2}\right)^2\right) \\ &= \frac{4}{9}\left(\frac{3x^2+3y^2}{2}\right) \\ &= \frac{2}{3}\left(x^2+y^2\right) \\ &= 156. \end{aligned}$$

Therefore, $AC^2 = x^2 + y^2 = \frac{3}{2} \cdot 156 = \boxed{234}$.

10. Three circles with radii 23, 46, and 69 are tangent to each other as shown in the figure below (figure is not drawn to scale).

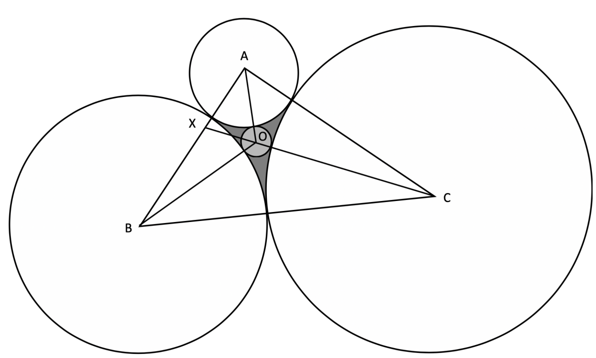


Find the radius of the largest circle that can fit inside the shaded region.

Answer: 6

Solution: Let A, B, C be the center of the three circles with radii $a = 23, b = 46, c = 69$ respectively and let O and r be the center and the radius of the largest circle that can fit in the shaded region. Hence, we have a triangle ABC with sides 69, 92, and 115 and a point O inside the triangle with distances $23 + r$, $46 + r$, and $69 + r$ from A, B, C respectively.

Suppose CO intersects AB at X and let $OX = x$ and $AX = y$.



Then, by Stewart's Theorem, we have

$$AO^2XC = AC^2OX + AX^2OC - (OX)(OC)(XC),$$
$$BO^2XC = BC^2OX + BX^2OC - (OX)(OC)(XC),$$
$$OX^2AB = OA^2XB + OB^2XA - (XB)(XA)(BA).$$

Substituing in the values,

$$(23 + r)^2(69 + r + x) = 92^2x + y^2(69 + r) - x(69 + r)(69 + r + x),$$
$$(46 + r)^2(69 + r + x) = 115^2x + (69 - y)^2(69 + r) - x(69 + r)(69 + r + x),$$
$$x^2(69) = (23 + r)^2(69 - y) + (46 + r)^2(y) - (69 - y)(y)(69).$$

Solving these equations such that $r > 0$ yields $x = \frac{125}{6}, y = \frac{161}{6}, r = \boxed{6}$.