1. If $f(x) = nx$, $g(x) = e^{2x}$, and $h(x) = g(f(x))$, find n such that $h'(0) = 100$.

   Answer: 50

   Solution: If $f(x) = nx$, $g(x) = e^{2x}$, and $h(x) = g(f(x))$, then $h(x) = e^{2(nx)} = e^{2nx}$. The derivative of $h(x)$ is $2n * e^{2nx}$, by the Chain Rule. Then, plugging in 0 for x gets us $2n = 100$, so, $n = 50$.

2. Farmer Joe will plant carrots to cover a rectangle in the first quadrant with a vertex at the origin and sides parallel to the x and y axes. However, he can not grow carrots on his neighbor's land. If the border between his and his neighbor's land is along the curve $y = -\ln(2x)$, what is the maximum area of carrotland Farmer Joe can create?

   Answer: $\mathbf{\frac{1}{2e}}$

   Solution: We note that the area of carrotland is $xy = -x\ln(2x)$. The maximum occurs when $(xy)' = 0$, or $-\ln(2x) + -1 = 0$. Hence $x = e^{-1}/2$ and $y = 1$. So, the maximum area is $\boxed{\frac{1}{2e}}$.

3. For all θ from 0 to 2π, Annie draws a line segment of length θ from the origin in the direction of θ radians. What is the area of the spiral swept out by the union of these line segments?

   Answer: $\mathbf{\frac{4\pi^3}{3}}$

   Solution: After drawing the spiral, it should become clear that we have the following calculation since our radius is θ

$$\frac{1}{2}\int_0^{2\pi} \theta^2 d\theta = \boxed{\frac{4\pi^3}{3}}$$

4. The Chebyshev Polynomials are defined as

$$T_n(x) = \cos(n\cos^{-1}(x)),$$

   for $n = 0, 1, 2, ...$ Compute the following infinite series:

$$\sum_{n=1}^{\infty} \int_{-1}^{1} T_{2n+1}(x)dx.$$

   If the series diverges, your answer should be "D."

   Answer: 0

   Solution: We can show that the Chebyshev Polynomials are odd for odd n. Recall that for an odd function, $f(-x) = -f(x)$. So, the integral of said function over $[-1, 1]$ should be 0. Thus, the sum of those integrals should also be 0.

5. What is

$$(2020)^2 + \frac{(2021)^2}{1!} + \frac{(2022)^2}{2!} + \frac{(2023)^2}{3!} + \frac{(2024)^2}{4!} + \dots$$

   Answer: $\mathbf{4084442}e$

   Solution: We start with

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Then we can do a pattern of differentiating and multiplying by x.

$$x^{2020}e^x = \sum_{n=0}^{\infty} \frac{x^{n+2020}}{n!}$$

$$(2020x^{2019} + x^{2020})e^x = \sum_{n=0}^{\infty} \frac{(n+2020)x^{n+2019}}{n!}$$

$$(2020x^{2020} + x^{2021})e^x = \sum_{n=0}^{\infty} \frac{(n+2020)x^{n+2020}}{n!}$$

$$(2020^2x^{2019} + 4041x^{2020} + x^{2021})e^x = \sum_{n=0}^{\infty} \frac{(n+2020)^2x^{n+2019}}{n!}$$

So, our desired sum occurs when $x = 1$, and we obtain $\boxed{4084442e}$.

6. Let us define the sequence $a_n = (-1)^n/(n)$. Now, we define the partial sums

$$A_N = \sum_{n=1}^{N} a_n.$$

What is the difference

$$\sum_{N=1}^{\infty}\left(A_N - \lim_{M\to\infty} A_M)\right)?$$

Answer: $-\log(2) + 1/2$

Solution: First we note that we are calculating the series

$$\sum_{N=1}^{\infty}\sum_{m=N+1}^{\infty} \frac{(-1)^{m+1}}{m} = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-1)^{n+m}}{n+m}$$

Instead, we consider

$$F(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(-x)^{n+m}}{n+m}.$$

Then we note that the answer should be $\lim_{x\to 1} -F(x)$. Now we can see that

$$F'(x) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(-1)(-x)^{n+m-1} = \frac{x}{(1+x)^2}.$$

Then we can determine that

$$F(x) = \int \frac{x}{(1+x)^2}dx = \log(1+x) + \frac{1}{1+x} - 1$$

(Note that $F(0) = 0$ from its definition). Thus,

$$\lim_{x\to 1} -F(x) = -\log(2) - \frac{1}{2} + 1 = \boxed{-\log(2) + 1/2.}$$

7. Define $f_1(x) = x$ and for every integer $n \geq 2$, we define $f_n(x) = x^{f_{n-1}(x)}$. Compute

$$\lim_{n\to\infty} \int_e^{2020} \frac{f_n'(x)}{f_n(x) f_{n-1}(x) \ln x} - \frac{f_{n-1}'(x)}{f_{n-1}(x)} dx.$$

Answer: $\ln(\ln 2020)$

Solution: It turns out that the limit is unnecessary, as we can see by induction that

$$f_n'(x) = f_n(x) \left(f_{n-1}'(x) \ln x + \frac{1}{x} f_{n-1}(x) \right).$$

This means that the desired integral is $\int_e^{2020} \frac{1}{x \ln x} dx$. The anti-derivative is just $\ln(\ln x)$, so evaluating at endpoints gives $\boxed{\ln(\ln 2020)}$.

8. Compute

$$\int_0^\infty \frac{dx}{x^4 - 6x^2 + 25}.$$

Answer: $\frac{\pi}{20}$

Solution: We first factor the denominator to get

$$\int_0^\infty \frac{dx}{x^4 - 6x^2 + 25} = \int_0^\infty \frac{dx}{(x^2 - 4x + 5)(x^2 + 4x + 5)}.$$

We can then decompose the integral into the partial fractions

$$\int_0^\infty \left[\frac{-x + 4}{40(x^2 - 4x + 5)} + \frac{x + 4}{40(x^2 - 4x + 5)} \right] dx.$$

Focusing on the first term, we notice that $\frac{d}{dx}(x^2 - 4x + 5) = 2x - 4$. This suggests that we further decompose the first term into

$$\int_0^\infty \left[\frac{-(x - 2)}{40(x^2 - 4x + 5)} + \frac{2}{40(x^2 - 4x + 5)} \right] dx.$$

The first integral evaluates to $-\frac{1}{2} \ln(x^2 - 4x + 5)$. To evaluate the second integral, we complete the square in the denominator to get

$$\int_0^\infty \frac{2dx}{40(x-2)^2 + 40}.$$

We can then make the substitution $u = x - 2$ and use the fact that $\int \frac{dx}{x^2+1} = \tan^{-1}(x)$ to see that the second integral evaluates to $\frac{1}{20} \tan^{-1}(x - 2)$. Decomposing the second integral in a similar manner, we find

$$\begin{aligned} \int_0^\infty \frac{dx}{x^4 - 6x^2 + 25} &= \Bigg[-\frac{1}{80} \ln(x^2 - 4x + 5) + \frac{1}{80} \ln(x^2 + 4x + 5) \\ &\quad + \frac{1}{20} \tan^{-1}(x - 2) + \frac{1}{20} \tan^{-1}(x + 2) \Bigg]_0^\infty \\ &= \left[\frac{1}{80} \ln\left(\frac{x^2 - 4x + 5}{x^2 + 4x + 5} \right) + \frac{1}{20} \left(\tan^{-1}(x - 2) + \tan^{-1}(x + 2) \right) \right]_0^\infty \end{aligned}$$

When $x = 0$, the resulting terms cancel to 0. When $x \to \infty$, the fraction in the ln term approaches 1, and $\ln 1 = 0$. On the other hand, $\tan^{-1}(x) \to \frac{\pi}{2}$, so our answer is $\frac{1}{20}\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \boxed{\frac{\pi}{20}}$.

9. Define $a_n = \underbrace{\sqrt{2 + \sqrt{2 + \sqrt{2 + \dots}}}}_{\text{n square roots}}$. For example $a_1 = \sqrt{2}$ and $a_2 = \sqrt{2 + \sqrt{2}}$. Find the value of

$$\lim_{n\to\infty} 4^n(2 - a_n).$$

Answer: $\boldsymbol{\frac{\pi^2}{4}}$

Solution: It is not hard to show by induction that $a_n = 2\cos(\pi/2^{n+1})$. Therefore,

$$4^n(2 - a_n) = 4^n\left(2 - \left(2 - 2\frac{\left(\frac{\pi}{2^{n+1}}\right)^2}{2!} + 2\frac{\left(\frac{\pi}{2^{n+1}}\right)^4}{4!} - \dots\right)\right) = \frac{\pi^2}{4} + O(1/4^n).$$

Thus, as $n \to \infty$, the limit approaches $\frac{\pi^2}{4}$.

10. Let

$$I_m = \int_0^{2\pi} \sin(x)\sin(2x)\cdots\sin(mx)dx.$$

Find the sum of all integers $1 \le m \le 100$ such that $I_m \neq 0$.

Answer: 1300

Solution: Analyzing the even/oddness of the function shows that odd m don't work, and analyzing $\pi - x$ vs x symmetry shows that $m \equiv 2 \mod 4$ doesn't work. But it is not obvious why $I_m \neq 0$ for every $m \equiv 0 \mod 4$. One could guess this is true and guess the corresponding answer, but we present a full proof here (and we present differently phrased reasons for why $m \not\equiv 0 \mod 4$ fail).

First, recall we can re-express $\sin(nx)$ as $\frac{1}{2i}(e^{inx} - e^{-inx})$, and note that I_m being nonzero means that we can ignore the coefficient of $\frac{1}{2i}$ and simply find the m for which the following is nonzero:

$$\int_0^{2\pi}(e^{ix} - e^{-ix})(e^{i2x} - e^{-i2x})\cdots(e^{imx} - e^{-imx})dx.$$

When we expand this product, each term contributes one exponential of the form $s_n e^{ixs_n n}$ where $s_n \in \{-1, +1\}$, yielding

$$\int_0^{2\pi} \sum_{s_n\in\{-1,+1\}} s_1 s_2 \dots s_m \exp\left(ix\sum_n s_n n\right)dx.$$

Rearranging the sums and integrals, this becomes

$$\sum_{s_n\in\{-1,+1\}} s_1 s_2 \dots s_m \int_0^{2\pi} \exp\left(ix\sum_n s_n n\right)dx.$$

Notice that if $\sum_n s_n n$ is some nonzero integer,

$$\int_0^{2\pi} \exp\left(ix\sum_n s_n n\right) dx = \frac{1}{i\sum_n s_n n}\left(\exp\left(2\pi i \sum_n s_n n\right) - 1\right) = 0.$$

However, if $\sum_n s_n n = 0$, then the integral is just $\int_0^{2\pi} 1 dx = 2\pi$. Therefore, we again ignore scaling coefficients 2π, and the expression that should be nonzero is

$$\sum_{\substack{s_n \in \{-1,+1\}, \\ \sum_n s_n n = 0}} s_1 s_2 \dots s_m.$$

Now, notice that $\sum_n s_n n \equiv \sum_n n \equiv m(m+1)/2 \mod 2$. So if $\sum_n s_n n = 0$, then we must have $m(m+1) \equiv 0 \mod 4$, i.e. m is either 0 or 3 modulo 4.

For a given tuple $S = (s_1, s_2, \dots, s_m)$ such that $\sum_n s_n n = 0$, let's split the indices into two sets: $P_S = \{n : s_n = +1\}$ and $N_S = \{n : s_n = -1\}$. Notice that $s_1 s_2 \dots s_m = (-1)^{|N_S|}$, so the desired quantity can be written as

$$\sum_S (-1)^{|N_S|}.$$

If $m \equiv 3 \mod 4$, then $|P_S| \equiv -|N_S| \mod 2$ since $|P_S| + |N_S| = m \equiv 1 \mod 2$. Moreover, notice that a valid S can be paired with the valid tuple $-S := (-s_1, -s_2, \dots, -s_m)$, for which $N_{-S} = P_S$ and hence $(-1)^{|N_{-S}|} + (-1)^{|N_S|} = (-1)^{-|N_S|} + (-1)^{|N_S|} = 0$. Clearly, every valid tuple is paired with exactly 1 distinct valid tuple, showing that the desired total sum is 0 if $m \equiv 3 \mod 4$.

So assume $m = 4k$ for some positive integer k. In this case, $|P_S| \equiv |N_S| \mod 2$, meaning that $(-1)^{|N_{-S}|} + (-1)^{|N_S|} = 2(-1)^{|N_S|}$. Therefore, we can view the tuples S and $-S$ as equivalent (we refer to them jointly as $\pm S$), and we can view the tuple $(P_{\pm S}, N_{\pm S})$ as just a partition $\Pi_{\pm S}$ of $\{1, 2, \dots, m\}$ into two sets. Since $|P_{\pm S}| \equiv |N_{\pm S}| \mod 2$, let us define a partition $\Pi_{\pm S}$'s parity to be equal to the parity of $|P_{\pm S}|$.

Let O be the set of $\pm S$ with odd $\Pi_{\pm S}$ and E be the set of $\pm S$ with even $\Pi_{\pm S}$. These sets are clearly finite, and the desired sum is proportional to $|E| - |O|$. If the desired total sum is nonzero, then $|O| \neq |E|$. We claim that there exists a non-surjective injection from O to E, which would imply $|O| < |E|$.

Consider an element $\pm S$ of O and its partition $\Pi_{\pm S}$ into two sets A, B such that WLOG $A = \{1, 2, \dots, x\} \cup A'$ and $B = \{x+1\} \cup B'$ where all elements of A' and B' are at least $x+2$ and $x > 1$. These conditions generally hold because when $m = 4k$, we know $m \geq 4$, so the partition cannot be $\{1\}, \{2, 3, \dots, m\}$ (this implies the existence of $\{1, 2, \dots, x\}$ in A with $x > 1$) and the partition containing 1 cannot be $\{1, 2, \dots, m\}$ (this implies that $x+1$ lies in the set without 1).

Now, consider $A^* = \{2, 3, \dots, x-1\} \cup \{x+1\} \cup A'$ and $B^* = \{1, x\} \cup B'$. It is easy to see that since we only swapped $\{1, x\}$ with $\{x+1\}$, this is a partition that leads to a valid choice of $\pm S^*$. Moreover, since $\Pi_{\pm S}$ was odd, we know $|B'|$ is odd and hence $|B^*|$ is even, implying that $\pm S^* \in E$. Thus, we have a valid map from O to E.

It is easy to see that this is an injection, but the condition that elements of B' are at least $x+2$ means that it is not a surjection: consider attempting to map to the element that partitions the indices into those equivalent to 0 or 3 mod 4, and those equivalent to 1 or 2 mod 4. For $m \geq 8$, this results in one set in the partition having $\{1, 4, 5, 8, \dots\}$, meaning $1, x, x+1$ are in the same side of the partition and is hence impossible to achieve under the map, and for $m = 4$, there is simply no $x + 1$.

The answer is then $\sum_{k=1}^{25} 4k = \boxed{1300}$.