1. Let $f(x)$ be a polynomial with integer coefficients such that $f(u) - f(v)$ divides $u^2 - v^2$ for all integers u and v. Given that $f(0) = 1$ and $f(1) = 2$, find the largest possible value of $f(50)$.

   Answer: 2501

   Solution: Substitute $v = 0$. Since $f(u) - 1 \mid u^2$ for all u, we know that $\deg(f(x)) \le 2$.

   Let $f(x) = ax^2 + bx + c$, then $f(0) = c = 1$ and $f(1) = a + b + c = 2$ or $a + b = 1$. We also have:

   $$\frac{f(u) - f(v)}{u - v} = \frac{a(u^2 - v^2) + b(u - v)}{u - v} = a(u + v) + b$$

   divides $u + v$ for all integers u and v. If $a \neq 0$ then b has to be 0, meaning that $a = 1$ and $f(x) = x^2 + 1$. If $a = 0$ then $b = 1$, $f(x) = x + 1$.

   Thus the largest possible value of $f(50)$ is $50^2 + 1 = \boxed{2501}$.

2. How many complex numbers z have the property that $z^2 = \bar{z}$, where $\bar{z}$ is the complex conjugate of z?

   Answer: 4

   Solution: If $z^2 = \bar{z}$, then $\left|z^2\right| = |\bar{z}| = |z|$ so $|z| = 0$ or $|z| = 1$. If $|z| = 0$, then $z = 0$ is the only solution. If $|z| = 1$, then $z = e^{i\theta}$ and $e^{2i\theta} = e^{-i\theta}$, so $e^{3i\theta} = 1$. The $\boxed{4}$ solutions are 0, 1, $e^{i\pi/3}$, and $e^{2i\pi/3}$.

3. Compute

   $$\sum_{n=1}^{\infty} \binom{n}{2} \left(\frac{3}{4}\right)^n.$$

   Answer: 36

   Solution 1: Let S be the sum we are trying to compute, and $f(x) = \sum_{n=1}^{\infty} n(n-1)x^n$. We can see that $S = \frac{1}{2} f(\frac{3}{4})$. Then we have:

   $$\begin{aligned} f(x) &= \sum_{n=2}^{\infty} n(n-1)x^n \\ &= \sum_{n=2}^{\infty} x^2 \cdot n(n-1)x^{n-2} \\ &= \sum_{n=2}^{\infty} x^2 \frac{d^2}{dx^2} x^{n-2} \\ &= x^2 \frac{d^2}{dx^2} \sum_{n=2}^{\infty} x^{n-2} \\ &= x^2 \frac{d^2}{dx^2} \frac{1}{1-x} \\ &= \frac{2x^2}{(1-x)^3} \end{aligned}$$

   Setting $x = \frac{3}{4}$, it is easy to see that $f(\frac{3}{4}) = 72$, and thus the desired sum is $\boxed{36}$.

   Solution 2: Define series A and B as follows:

$$
\begin{aligned}
A &= \sum_{n=1}^{\infty} n\left(\frac{3}{4}\right)^n \\
B &= \sum_{n=1}^{\infty} n(n-1)\left(\frac{3}{4}\right)^n \\
\frac{3}{4}A &= \sum_{n=2}^{\infty} (n-1)\left(\frac{3}{4}\right)^n \\
A - \frac{3}{4}A &= \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n = \frac{3}{4}\cdot 4 = 3 \\
A &= 12 \\
\frac{3}{4}B &= \sum_{n=2}^{\infty} (n-1)(n-2)\left(\frac{3}{4}\right)^n \\
B - \frac{3}{4}B &= \sum_{n=2}^{\infty} 2(n-1)\left(\frac{3}{4}\right)^n \\
\frac{1}{4}B &= \frac{3}{4}\sum_{n=1} 2n\left(\frac{3}{4}\right)^n = \frac{3}{4}\cdot 2A = 18 \\
B &= 72
\end{aligned}
$$

The desired sum is equal to $B/2$, so the answer is $\boxed{36}$.