Time limit: 80 minutes.
Instructions: For this test, you work in teams of eight to solve 15 short answer questions and 5 proof questions.
No calculators.

Short Answer Questions: For the short answer questions, all answers must be expressed in simplest form unless specified otherwise. Submit a single answer sheet for grading. Only answers written inside the boxes on the answer sheet will be considered for grading.

1. Suppose A and B are points in the plane lying on the parabola $y = x^2$, and the x-coordinates of A and B are -29 and 51, respectively. Let C be the point where line AB intersects the y-axis. What is the y-coordinate of C?

2. Cindy has a collection of identical rectangular prisms. She stacks them, end to end, to form 1 longer rectangular prism. Suppose that joining 11 of them will form a rectangular prism with 3 times the surface area of an individual rectangular prism. How many will she need to join end to end to form a rectangular prism with 9 times the surface area?

3. A lattice point is a point (a, b) on the Cartesian plane where a and b are integers. Compute the number of lattice points in the interior and on the boundary of the triangle with vertices at $(0, 0)$, $(0, 20)$, and $(18, 0)$.

4. Let $1 = a_1 < a_2 < a_3 < ... < a_k = n$ be the positive divisors of n in increasing order. If $n = a_3^3 - a_2^3$, what is n?

5. A point (x_0, y_0) with integer coordinates is a primitive point of a circle if for some pair of integers (a, b), the line $ax + by = 1$ intersects the circle at (x_0, y_0). How many primitive points are there of the circle centered at $(2, -3)$ with radius 5?

6. Three distinct points are chosen uniformly at random from the vertices of a regular 2018-gon. What is the probability that the triangle formed by these points is a right triangle?

7. Consider any 5 points placed on the surface of a cube of side length 2 centered at the origin. Let m_x be the minimum distance between the x coordinates of any of the 5 points, m_y be the minimum distance between y coordinates, and m_z be the minimum distance between z coordinates. What is the maximum value of $m_x + m_y + m_z$?

8. Eddy has two blank cubes A and B and a marker. Eddy is allowed to draw a total of 36 dots on cubes A and B to turn them into dice, where each side has an equal probability of appearing, and each side has at least one dot on it. Eddy then rolls dice A twice and dice B once and computes the product of the three numbers. Given that Eddy draws dots on the two dice to maximize his expected product, what is his expected product?

9. Let $ABCD$ be a square. Point E is chosen inside the square such that $AE = 6$. Point F is chosen outside the square such that $BE = BF = 2\sqrt{5}$, $\angle ABF = \angle CBE$, and $AEBF$ is cyclic. Compute the area of $ABCD$.

10. Find the total number of sets of nonnegative integers (w, x, y, z) where $w \leq x \leq y \leq z$ such that $5w + 3x + y + z = 100$.

11. Let $f(k)$ be a function defined by the following rules:

    (a) $f(k)$ is multiplicative. In other words, if $\gcd(a, b) = 1$, then $f(ab) = f(a) \cdot f(b)$,

    (b) $f(p^k) = k$ for primes $p = 2, 3$ and all $k > 0$,

    (c) $f(p^k) = 0$ for primes $p > 3$ and all $k > 0$, and

(d) $f(1) = 1$.

For example, $f(12) = 2$ and $f(160) = 0$. Evaluate

$$\sum_{k=1}^{\infty} \frac{f(k)}{k}.$$

12. Consider all increasing arithmetic progressions of the form $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ such that $a, b, c \in \mathbb{N}$ and $\gcd(a, b, c) = 1$. Find the sum of all possible values of $\frac{1}{a}$.

13. In $\triangle ABC$, let D, E, and F be the feet of the altitudes drawn from A, B, and C respectively. Let P and Q be points on line EF such that BP is perpendicular to EF and CQ is perpendicular to EF. If $PQ = 2018$ and $DE = DF + 4$, find DE.

14. Let A and B be two points chosen independently and uniformly at random inside the unit circle centered at O. Compute the expected area of $\triangle ABO$.

15. Suppose that a, b, c, d are positive integers satisfying

$$\begin{aligned} 25ab + 25ac + b^2 &= 14bc \\ 4bc + 4bd + 9c^2 &= 31cd \\ 9cd + 9ca + 25d^2 &= 95da \\ 5da + 5db + 20a^2 &= 16ab \end{aligned}$$

Compute $\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}$.

Proof Questions: For the proof questions, answers should be written on sheets of scratch paper, clearly labeled, with every problem *on its own sheet.* If you have multiple pages for a problem, number them and write the total number of pages for the problem (e.g. 1/2, 2/2).

1. Prove that if 7 divides $a^2 + b^2 + 1$, then 7 does not divide $a + b$.

2. Consider a game played on the integers in the closed interval $[1, n]$. The game begins with some tokens placed in $[1, n]$. At each turn, tokens are added or removed from $[1, n]$ using the following rule: For each integer $k \in [1, n]$, if exactly one of $k - 1$ and $k + 1$ has a token, place a token at k for the next turn, otherwise leave k blank for the next turn.

   We call a position static if no changes to the interval occur after one turn. For instance, the trivial position with no tokens is static because no tokens are added or removed after a turn (because there are no tokens). Find all non-trivial static positions.

3. Show that if A is a shape in the Cartesian coordinate plane with area greater than 1, then there are distinct points $(a, b), (c, d)$ in A where $a - c = 2x + 5y$ and $b - d = x + 3y$ where x, y are integers.

4. Let F_k denote the series of Fibonacci numbers shifted back by one index, so that $F_0 = 1$, $F_1 = 1$, and $F_{k+1} = F_k + F_{k-1}$. It is known that for any fixed $n \geq 1$ there exist real constants b_n, c_n such that the following recurrence holds for all $k \geq 1$:

$$F_{n\cdot(k+1)} = b_n \cdot F_{n\cdot k} + c_n \cdot F_{n\cdot(k-1)}.$$

   Prove that $|c_n| = 1$ for all $n \geq 1$.

5. Let $ABCD$ be a quadrilateral with sides AB, BC, CD, DA and diagonals AC, BD. Suppose that all sides of the quadrilateral have length greater than 1, and that the difference between any side and diagonal is less than 1. Prove that the following inequality holds:

$$(AB + BC + CD + DA + AC + BD)^2 > 2|AC^3 - BC^3| + 2|BD^3 - AD^3| - (AB + CD)^3$$