1. A circle of radius 2 is inscribed in equilateral triangle ABC. The altitude from A to BC intersects the circle at a point D not on BC. BD intersects the circle at a point E distinct from D. Find the length of BE.

   Answer: $\frac{6}{\sqrt{7}}$

   Solution: We can find the side length of the triangle as follows: Since the triangle is equilateral, the segment from the center of the circle to a vertex (say A) bisects the angle. Also, the segment from the center to an adjacent point of tangency (say the perpendicular to AC) creates a right angle. So we get a 30-60-90 triangle, which tells us that the side length of the triangle is $4\sqrt{3}$.

   Next, we call the point of intersection of AD and BC point F. Consider triangle DBF. Using the Pythagorean theorem, $BD = 2\sqrt{7}$. By power of a point, $BE \cdot BD = BF^2$. So $BE = \frac{BF^2}{BD} = \frac{12}{2\sqrt{7}} = \boxed{\frac{6}{\sqrt{7}}}$.

2. Points A, B, and C lie on a circle of radius 5 such that $AB = 6$ and $AC = 8$. Find the smaller of the two possible values of BC.

   Answer: $\frac{14}{5}$

   Solution: Fix segment AB, and let C and D be the two points on the circle 8 units from A, where C is closer to B than D. Observe that BD is a diameter (and hence $BD = 10$) because a 6-8-10 inscribed right triangle must be possible.

   Next, we see that CD can be calculated by drawing the diameter that goes through A, intersecting the opposite side of the circle at point E. Note that $CD \perp AE$. Moreover, ACE and ADE are right, since they are inscribed in semicircles, and $EC = ED = 6$ by the Pythagorean Theorem. Computing the area of quadrilateral $ACED$ two different ways, we get

$$\frac{1}{2} \cdot AE \cdot CD = 5 \cdot CD = \frac{1}{2} \cdot AC \cdot CE + \frac{1}{2} \cdot AD \cdot DE = 48 \implies CD = \frac{48}{5}.$$

   Finally, since BCD is a right triangle with $CD = \left(\frac{2}{5}\right) 24$ and $BD = \left(\frac{2}{5}\right) 25$, we conclude that $BC = \left(\frac{2}{5}\right) 7 = \boxed{\frac{14}{5}}$.

3. In quadrilateral $ABCD$, diagonals AC and BD intersect at E. If $AB = BE = 5$, $EC = CD = 7$, and $BC = 11$, compute AE.

   Answer: $\frac{47}{7}$

   Solution 1: First, notice that length AE is completely determined by the fact that $AB = BE$ and by the lengths of AB, BC and EC. Thus, we only consider the triangle ABC. First, drop altitude BH and note that since ABE is isoceles, $EH = \frac{1}{2}AE$. Now, using Pythagoras twice, we have

$$BH^2 = 5^2 - EH^2$$
$$BH^2 = 11^2 - (7 + EH)^2.$$

   Setting these two equations to be equal, we can thus solve the equation $25 = 72 - 14EH$. Therefore, $AE = 2EH = \boxed{\frac{47}{7}}$.

Solution 2: Since $\angle AEB \cong \angle DEC$, we have $\triangle AEB \sim \triangle DEC$ by SAS. Hence, $\frac{AE}{DE} = \frac{BE}{CE} \implies \frac{AE}{BE} = \frac{DE}{CE}$. Additionally, $\angle BEC \cong \angle AED$, so $\triangle BEC \sim \triangle AED$ by SAS again.

Now, we do some angle-chasing. Since $\angle BAE \cong \angle CDE$ and $\angle EAD \cong \angle EBC$, $\angle ABC$ and $\angle ADC$ are supplementary. Hence, $ABCD$ is cyclic.

Let $AE = 5x$, so $DE = 7x$ since $\frac{AE}{DE} = \frac{BE}{CE} = \frac{5}{7}$. Also, note that $AD = x \cdot BC = 11x$ because $\frac{AD}{BC} = \frac{AE}{BE} = \frac{5x}{5} = x$.

Now, Ptolemy's Theorem gives us

$$\begin{aligned} 5 \cdot 7 + 11 \cdot 11x &= (5x+7)(7x+5) \\ \implies 121x + 35 &= 35x^2 + 74x + 35 \\ \implies 121x &= 35x^2 + 74x \\ \implies 47x &= 35x^2 \\ \implies x &= \frac{47}{35} \end{aligned}$$

because $x > 0$.

Hence, report $5x = \boxed{\frac{47}{7}}$.