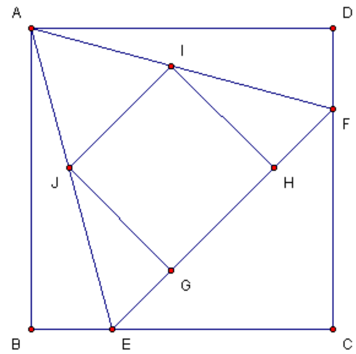
1. Let $ABCD$ be a unit square. The point E lies on BC and F lies on AD. $\triangle AEF$ is equilateral. $GHIJ$ is a square inscribed in $\triangle AEF$ so that $\overline{GH}$ is on $\overline{EF}$. Compute the area of $GHIJ$.



Answer: $312 - 180\sqrt{3}$

First let a be the length of AE. Then $CE = a/\sqrt{2}$, $BE = 1 - a/\sqrt{2}$ so $AE^2 = a^2 = 1 + BE^2 = 2 - \sqrt{2}a + a^2/2$. Solving it gives $a^2 + 2\sqrt{2}a - 4 = 0$, $(a+\sqrt{2})^2 = 6$ so $a = \sqrt{6} - \sqrt{2}$.
Next let b be the length of IJ. Then AIJ is equilateral so $AJ = b$. Also $JE = 2/\sqrt{3}b$, so $AE = a = \frac{2+\sqrt{3}}{\sqrt{3}}b$, $b = (2-\sqrt{3})(\sqrt{3})(\sqrt{6}-\sqrt{2}) = \sqrt{2}(9-5\sqrt{3})$. Squaring it gives $312 - 180\sqrt{3}$.

2. Find all integers x for which $|x^3 + 6x^2 + 2x - 6|$ is prime.

Answer: 1, −1

The whole equation is $\equiv 0 \pmod 3$, so $x^3 + 6x^2 + 2x - 6$ should be 3 or -3. The equation $(x^3 + 6x^2 + 2x - 6)^2 = 3^2$ can be rewritten using difference of squares as $(x-1)(x^2+7x-9)(x+1)(x^2+5x-3) = 0$, so only 1 and -1 work for x.

3. Let A be the set of points (a, b) with $2 < a < 6, -2 < b < 2$ such that the equation

$$ax^4 + 2x^3 - 2(2b - a)x^2 + 2x + a = 0$$

has at least one real root. Determine the area of A.

Answer: 12

After dividing the equation by $4x^2$, we can re-write it as

$$a\left(\frac{x}{2} + \frac{1}{2x}\right)^2 + \left(\frac{x}{2} + \frac{1}{2x}\right) - a = b.$$

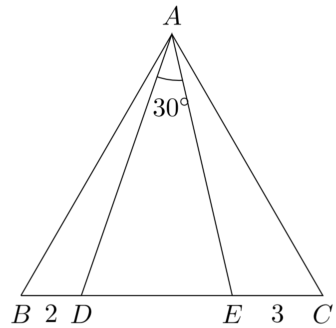
Set $y = \frac{x}{2} + \frac{1}{2x}$, which has range $(-\infty, -1] \cup [1, \infty)$. Therefore, we need all b in $(-2, 2)$ such that b is in the range of $f(y) = ay^2 + y - a$ for the domain $y \in (-\infty, -1] \cup [1, \infty)$. The vertex of this parabola lies at $y = -\frac{1}{2a} \in (-1/4, -1/12)$, so the desired range is just all values greater than $f(-1) = -1$. Hence, A is the set of all points where $-1 < b < 2$ and $2 < a < 6$, so the area is 12.

4. Three nonnegative reals x, y, z satisfy $x + y + z = 12$ and $xy + yz + zx = 21$. Find the maximum of xyz.

Answer: 10

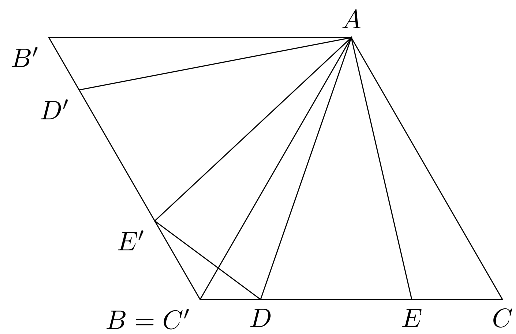
Consider the graphs of $y = t^3 - 12t^2 + 21t$ and $y = p (p \leq 0)$. These two graphs intersect at three points (counting multiplicity) if and only if there are three nonnegative x, y, z satisfying $xyz = p$. In order for these two to intersect at three points, p should lie between the local maximum and the local minimum of the cubic function $y = t^3 - 12t^2 + 21t$, so the maximal p will lie at the local maximum of this cubic. Since $y' = 3t^2 - 24t + 21 = 3(t-1)(t-7)$, the local maximum occurs at $t = 1$, so the local maximum is $1^3 - 12 \cdot 1^2 + 21 \cdot 1 = \boxed{10}$ (this can be achieved by letting $(x, y, z) = (1, 1, 10)$).

5. Let $\triangle ABC$ be equilateral. Two points D and E are on side BC (with order B, D, E, C), and satisfy $\angle DAE = 30^\circ$. If $BD = 2$ and $CE = 3$, what is BC?



Answer: $\mathbf{5 + \sqrt{19}}$

Rotate the figure around A by 60° so that C coincides with B. Let B', C', D', E' be the points corresponding to B, C, D, E in the rotated figure. Since $\angle E'AD = \angle E'AC' + \angle C'AD = \angle EAC + \angle BAD = 30^\circ = \angle EAD$, $E'A = EA$ and $DA = D'A$, one has $E'D = ED$. So $BC = BD + DE + EC$ can be found if we know $E'D$. But $E'D = \sqrt{E'B^2 + BD^2 - 2 \cdot E'B \cdot BD \cdot \cos 120^\circ} = \sqrt{19}$, so $BC = 2 + \sqrt{19} + 3 = 5 + \sqrt{19}$.



6. Three numbers are chosen at random between 0 and 2. What is the probability that the difference between the greatest and least is less than $\frac{1}{4}$?

Answer: $\mathbf{\frac{11}{256}}$

Call the three numbers x, y, and z. By symmetry, we need only consider the case $2 \geq x \geq y \geq z \geq 0$. Plotted in 3D, the values of (x, y, z) satisfying these inequalities form a triangular pyramid with a leg-2 right isosceles triangle as its base and a height of 2, with a volume of $2 \cdot 2 \cdot \frac{1}{2} \cdot 2 \cdot \frac{1}{3} = \frac{4}{3}$. We now need the volume of the portion of the pyramid satisfying $x - z \leq \frac{1}{4}$. The equation $z = x - \frac{1}{4}$ is a plane which slices off a skew triangular prism along with a small triangular pyramid at one edge of the large triangular pyramid. The prism has a leg-$\frac{1}{4}$ right isosceles triangle as its base and a height of $\frac{7}{4}$, so has volume $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{7}{4} = \frac{7}{2^7}$. The small triangular pyramid also has a leg-$\frac{1}{4}$ right isosceles triangle as its base and a height of $\frac{1}{4}$, so has volume $\frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{3 \cdot 2^7}$. Then our probability is $\left(\frac{7}{2^7} + \frac{1}{3 \cdot 2^7}\right) / \left(\frac{4}{3}\right) = 11/256$.

7. Tony the mouse starts in the top left corner of a 3x3 grid. After each second, he randomly moves to an adjacent square with equal probability. What is the probability he reaches the cheese in the bottom right corner before he reaches the mousetrap in the center?

Answer: $\mathbf{\frac{1}{7}}$

Let x be the probability that Tony reaches the cheese before the mousetrap, starting from the top left. Let y be the probability that Tony reaches the cheese before the mousetrap, starting from the top right or the bottom left (which are symmetric).

After 2 moves from the top left there is $\frac{1}{3}$ chance that Tony returns to the top left corner, there is $\frac{1}{3}$ chance that Tony reaches the mousetrap, and there is $\frac{1}{3}$ chance that Tony reaches the top right or bottom left corners. This gives us the relation

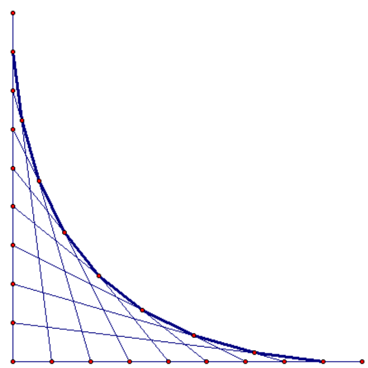
$$x = \frac{1}{3}x + \frac{1}{3}0 + \frac{1}{3}y.$$

After 2 moves from the top right corner there is $\frac{1}{3}$ chance that Tony returns to the top right corner, $\frac{1}{3}$ chance that Tony reaches the mousetrap, $\frac{1}{6}$ chance that Tony reaches the top left corner, and $\frac{1}{6}$ chance that Tony reaches the cheese. This gives the relation

$$y = \frac{1}{3}y + \frac{1}{3}0 + \frac{1}{6}x + \frac{1}{6}.$$

Now we have a system of linear of equations and we solve, obtaining $x = \frac{1}{7}$.

8. Let $A = (0, 0)$, $B = (1, 0)$, and $C = (0, 1)$. Divide AB into n equal segments, and call the endpoints of these segments $A = B_0, B_1, B_2, \cdots, B_n = B$. Similarly, divide AC into n equal segments with endpoints $A = C_0, C_1, C_2, \cdots, C_n = C$. By connecting B_i and C_{n-i} for all $0 \le i \le n$, one gets a piecewise curve consisting of the uppermost line segments. Find the equation of the limit of this piecewise curve as n goes to infinity.



Answer: $\sqrt{x} + \sqrt{y} = 1$ or equivalent form

The limiting curve is the boundary of a region given by the union of all line segments connecting $(q, 0)$ and $(0, 1-q)$ for all numbers $0 \le q \le 1$. Such a line segment has equation $\frac{x}{q} + \frac{y}{1-q} = 1$. Thus a point (x_0, y_0) is in that region if and only if the equation $\frac{x}{q} + \frac{y}{1-q} = 1$, $(1-q)x + qy = q(1-q)$ has a solution in $0 \le q \le 1$. Let $F(q) = (1-q)x + qy - q(1-q) = q^2 - (1 + x - y)q + x$. Note that $F(0) = x \ge 0$ and $F(1) = y \ge 0$, and the minimum of F at $\frac{1+x-y}{2}$ is always between 0 and 1. So F has a root in $[0, 1]$ if and only if $F(\frac{1+x-y}{2}) = -\frac{(1+x-y)^2}{4} + x \le 0$. So $4x \le (1 + x - y)^2$, $2\sqrt{x} \le 1 + x - y$, $y \le 1 - 2\sqrt{x} + x = (1 - \sqrt{x})^2$, $\sqrt{y} \le 1 - \sqrt{x}$, and finally we have $\sqrt{x} + \sqrt{y} \le 1$.

9. Determine the maximum number of distinct regions into which 2011 circles of arbitrary size can partition the plane.

Answer: $2011^2 - 2011 + 2 = 4042112$

Let $f(n)$ denote the maximum number of regions into which n circles can partition the plane. We will show that $f(n)$ is a quadratic polynomial in n. Indeed, let A be a planar arrangement of n circles. Note that A is a graph: Each intersection point is a vertex, and the arcs connecting them are edges. Having recognized this, we can apply Euler's theorem, $V - E + F = 2$, to find the number of regions (i.e., F). It is easy to see that an arrangement with the maximum number of vertices is optimal. The maximum number of vertices is $V = 2\binom{n}{2} = n(n-1)$, since each circle can intersect each other circle in at most two vertices. In this optimal arrangement, each circle contains $2(n-1)$ vertices and the same number of edges; thus, the total number of edges is $E = 2n(n-1)$. Thus, the desired quantity is $f(n) = E - V + 2 = n^2 - n + 2$, so our answer is $2011^2 - 2011 + 2 = 4042112$.

Alternative Solution: As before, we apply Euler's theorem for planar graphs. Given that circles are defined by quadratic polynomials, it is clear that V and E are each quadratic in n. In particular, Euler's theorem implies that F is quadratic in n. Moreover, it is easy to check that $f(1) = 2$, $f(2) = 4$, and $f(3) = 8$. Interpolating gives $f(n) = n^2 - n + 1$, as in the first solution.

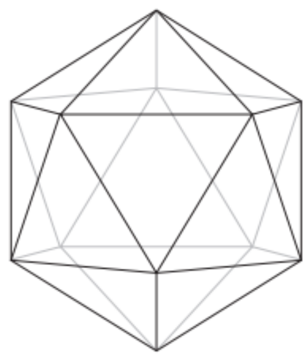
10. For positive reals x, y, and z, compute the maximum possible value of $\dfrac{xyz(x+y+z)}{(x+y)^2(y+z)^2}$.

Answer: $\mathbf{\frac{1}{4}}$

If we consider the triangle ABC with side length $AB = x + y$, $BC = y + z$, $CA = z + x$, the equation becomes

$$\frac{|ABC|^2}{AB^2 \cdot BC^2} = \frac{\sin^2 B}{4} \le \boxed{\frac{1}{4}}.$$

11. Find the diameter of an icosahedron with side length 1 (an icosahedron is a regular polyhedron with 20 identical equilateral triangle faces; a picture is given below).



Answer: $\mathbf{\frac{\sqrt{10+2\sqrt{5}}}{4}}$

Note that opposite vertices of the icosahedron can be seen as vertices of pyramids whose bases are regular pentagons of side length 1. Now, note that if we select two parallel diagonals of these two pentagons, these two diagonals are also two sides of a rectangle whose other sides are length 1 and whose diagonals are diameters of the icosahedron. The diagonal of the pentagon can be found with similar triangles: in regular pentagon $ABCDE$, let AD and BE intersect at F. Angle chasing shows that $\triangle ACD \sim \triangle DEF$, both are isosceles, and $FE = FA$, so we get that $\frac{AD}{1} = \frac{1}{AD-1} \implies AD = \frac{1+\sqrt{5}}{2}$. Hence, the diameter of the icosahedron equals $\sqrt{1^2 + (\frac{1+\sqrt{5}}{2})^2} = \frac{\sqrt{10+2\sqrt{5}}}{2}$.

12. Find the boundary of the projection of the sphere $x^2 + y^2 + (z-1)^2 = 1$ onto the plane $z = 0$ with respect to the point $P = (0, -1, 2)$. Express your answer in the form $f(x, y) = 0$, where $f(x, y)$ is a function of x and y.

Answer: $\mathbf{x^2 - 4y - 4 = 0}$

Let $O = (0, 0, 1)$ be the center of the sphere. For a point $X = (x, y, 0)$ on the boundary of the projection, the angle $\angle XPO$ is constant as X varies, since it is just the angle between OP and any tangent from P to the sphere. Considering the case when $X = (0, -1, 0)$, we can see that $\angle XPO = 45°$. Writing this in terms of the dot product, one has $(\overrightarrow{PO} \cdot \overrightarrow{PX})^2 = \frac{1}{2}|\overrightarrow{PO}|^2|\overrightarrow{PX}|^2$, which is equivalent to $((0, 1, -1) \cdot (x, y+1, -2))^2 = \frac{1}{2}|(0, 1, -1)|^2|(x, y+1, -2)|^2$, or $(y+3)^2 = x^2 + (y+1)^2 + 4$. The answer is $x^2 - 4y - 4 = 0$.

13. Compute the number of pairs of 2011-tuples $(x_1, x_2, ..., x_{2011})$ and $(y_1, y_2, ..., y_{2011})$ such that $x_k = x_{k-1}^2 - y_{k-1}^2 - 2$ and $y_k = 2x_{k-1}y_{k-1}$ for $1 \le k \le 2010$, $x_1 = x_{2011}^2 - y_{2011}^2 - 2$, and $y_1 = 2x_{2011}y_{2011}$.

Answer: $\mathbf{2^{2011}}$

Define $z_k = x_k + iy_k$. Then the equations are equivalent to $z_{k+1} = {z_k}^2 - 2$, $z_{2012} = z_1$. Let α be a solution of $z_1 = \alpha + \alpha^{-1}$ (which always has two distinct solutions unless $z_1 = 2$ or -2). Then one can check by induction that $z_k = \alpha^{2^{k-1}} + \alpha^{-2^{k-1}}$. Since one has $z_{2012} = z_1$, $\alpha^{2^{2011}} + \alpha^{-2^{2011}} = \alpha + \alpha^{-1}$.

Set $N = 2^{2011}$ and rewrite the above as $\alpha^{2N} + 1 = \alpha^{N-1} + \alpha^{N+1}$, or $(\alpha^{N+1} - 1)(\alpha^{N-1} - 1) = 0$. Since N is even, $N + 1$ and $N - 1$ are relatively prime. So the equations $X^{N+1} = 1$ and $X^{N-1} = 1$ have only the root 1 in common. Therefore there are $(N + 1) + (N - 1) - 1 = 2N - 1$ possibilities for α. Meanwhile, any one value of $z_1 = \alpha + \alpha^{-1}$ corresponds to two choices of α except when $\alpha = 1$ or -1. So our $2N - 2$ choices of $\alpha \neq 1$ together give $N - 1$ different solutions for z_1, and $\alpha = 1$ give a single solution $z = 2$. The answer is $N = 2^{2011}$.

14. Compute $I = \displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1} dx$.

Answer: $\boldsymbol{\frac{\pi \ln(2)}{8}}$

Let I denote the integral we wish to compute. The function $f(x) = \dfrac{\ln(x+1)}{x^2+1}$ does not have an elementary antiderivative. We will use Taylor series to compute I. We can find the Taylor series for the function $\dfrac{\ln(x+1)}{x^2+1}$ using the following formulas:

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$
$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - \dots$$

These formulas aren't good everywhere, but they do hold in $(0, 1)$. We have

$$\begin{aligned} f(x) &= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right)\left(1 - x^2 + x^4 - x^6 + \dots\right) \\ &= x + \left(-\frac{1}{2}\right)x^2 + \left(\frac{1}{3} - 1\right)x^3 + \left(-\frac{1}{4} + \frac{1}{2}\right)x^4 + \left(\frac{1}{5} - \frac{1}{3} + 1\right)x^5 + \dots \end{aligned}$$

In particular, an antiderivative is given by

$$F(x) = \frac{1}{2}x^2 + \frac{1}{3}\left(-\frac{1}{2}\right)x^3 + \frac{1}{4}\left(\frac{1}{3} - 1\right)x^4 + \frac{1}{5}\left(-\frac{1}{4} + \frac{1}{2}\right)x^5 + \frac{1}{6}\left(\frac{1}{5} - \frac{1}{3} + 1\right)x^6 + \dots$$

The definite integral I is given by $F(1)$, i.e., the sum

$$I = \frac{1}{2} + \frac{1}{3}\left(-\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{3} - 1\right) + \frac{1}{5}\left(-\frac{1}{4} + \frac{1}{2}\right) + \frac{1}{6}\left(\frac{1}{5} - \frac{1}{3} + 1\right) + \dots$$

Now we use the facts that

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots = \frac{\pi}{4}$$
$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln(2),$$

from the Taylor series for $\tan^{-1}(x)$ and $\ln(x + 1)$ respectively. Notice that in the above sum, every number of the form $\dfrac{1}{r \cdot s}$, where r is even and s is odd, occurs exactly once, with a positive sign if $r + s \equiv 3 \pmod 4$ and a negative sign if $1 \pmod 4$. Therefore, it is clear that

$$\begin{aligned} I &= \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)\left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \dots\right) \\ &= \frac{\pi}{4} \cdot \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right) \\ &= \frac{\pi \ln(2)}{8}. \end{aligned}$$

15. Find the smallest $\alpha > 0$ such that there exists $m > 0$ making the following equation hold for all positive integers $a, b \geq 2$:
$$\left(\frac{1}{\gcd(a, b-1)} + \frac{1}{\gcd(a-1, b)}\right)(a+b)^\alpha \geq m.$$

Answer: $\mathbf{\frac{1}{2}}$

Note that both $\gcd(a, b-1)$ and $\gcd(a-1, b)$ divide $a+b-1$. Also they are relatively prime, since $\gcd(a, b-1) \mid a$ and $\gcd(a-1, b) \mid a-1$. So their product is less than or equal to $a+b-1$, and therefore by the AM-GM inequality we have

$$\frac{1}{\gcd(a, b-1)} + \frac{1}{\gcd(a-1, b)} \geq 2\sqrt{\frac{1}{\gcd(a, b-1) \cdot \gcd(a-1, b)}} \geq \frac{2}{\sqrt{a+b-1}}.$$

Thus $\alpha = \frac{1}{2}$ and $m = 2$ suffice. To show that there is no such m for smaller α, let $b = (a-1)^2$. Then $\gcd(a, b-1) = a$ and $\gcd(a-1, b) = a-1$, so

$$\left(\frac{1}{\gcd(a, b-1)} + \frac{1}{\gcd(a-1, b)}\right)(a+b)^\alpha = \frac{(2a-1)(a^2-a+1)^\alpha}{a(a-1)}$$

and the limit when a goes to ∞ is zero if $\alpha < \frac{1}{2}$.