1. Let $F(x)$ be a real-valued function defined for all real $x \neq 0, 1$ such that

$$F(x) + F\left(\frac{x-1}{x}\right) = 1 + x.$$

Find $F(2)$.

Answer: $\mathbf{\frac{3}{4}}$

Setting $x = 2$, we find that $F(2) + F\left(\frac{1}{2}\right) = 3$. Now take $x = \frac{1}{2}$, to get that $F\left(\frac{1}{2}\right) + F(-1) = \frac{3}{2}$. Finally, setting $x = -1$, we get that $F(-1) + F(2) = 0$. Then we find that

$$\begin{aligned} F(2) &= 3 - F\left(\frac{1}{2}\right) = 3 - \left(\frac{3}{2} - F(-1)\right) = \frac{3}{2} + F(-1) = \frac{3}{2} - F(2) \\ \Rightarrow F(2) &= \frac{3}{4}. \end{aligned}$$

Alternate Solution: We can explicitly solve for $F(x)$ and then plug in $x = 2$. Notice that for $x \neq 0, 1$, $F(x) + F\left(\frac{x-1}{x}\right) = 1 + x$ so

$$F\left(\frac{x-1}{x}\right) + F\left(\frac{1}{1-x}\right) = 1 + \frac{x-1}{x} \text{ and } F\left(\frac{1}{1-x}\right) + F(x) = 1 + \frac{1}{1-x}.$$

Thus

$$\begin{aligned} 2F(x) &= F(x) + F\left(\frac{x-1}{x}\right) - F\left(\frac{x-1}{x}\right) - F\left(\frac{1}{1-x}\right) + F\left(\frac{1}{1-x}\right) + F(x) \\ &= 1 + x - \left(1 + \frac{x-1}{x}\right) + 1 + \frac{1}{1-x} \\ &= 1 + x + \frac{1-x}{x} + \frac{1}{1-x}. \end{aligned}$$

It follows that $F(x) = \frac{1}{2}\left(1 + x + \frac{1-x}{x} + \frac{1}{1-x}\right)$ and the result follows by taking $x = 2$.

2. Given that $a_1 = 2$, $a_2 = 3$, $a_n = a_{n-1} + 2a_{n-2}$, what is $a_{100} + a_{99}$?

Answer: $\mathbf{2^{98} \times 5}$

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-2} \\ a_n + a_{n-1} &= 2(a_{n-1} + a_{n-2}) \\ &= 2^{n-2}(a_1 + a_2). \end{aligned}$$

So $a_{100} + a_{99} = 2^{98} \times 5$.

3. Let sequence A be $\{\frac{7}{4}, \frac{7}{6}, \frac{7}{9}, \ldots\}$ where the j^{th} term is given by $a_j = \frac{7}{4}\left(\frac{2}{3}\right)^{j-1}$. Let B be a sequence where the j^{th} term is defined by $b_j = a_j^2 + a_j$. What is the sum of all the terms in B?

Answer: $\mathbf{\frac{861}{80}}$

Split B into two series C and D where the terms of C are $c_j = a_j^2 = \frac{49}{16}\left(\frac{4}{9}\right)^{j-1}$ and the terms of D are $d_j = a_j = \frac{7}{4}\left(\frac{2}{3}\right)^{j-1}$. Since both C and D are geometric series with ratios less than 1, the sum of their terms yields $\frac{49/16}{1-4/9} = \frac{441}{80}$ and $\frac{7/4}{1-2/3} = \frac{21}{4}$. Therefore, the sum of the terms in B equals $\frac{441}{80} + \frac{21}{4} = \frac{861}{80}$.

4. Find all rational roots of $|x - 1| \times |x^2 - 2| - 2 = 0$.

Answer: $\boldsymbol{x = -1, 0, 2}$

There are four intervals to consider, each with their own restrictions. Consider the case in which $x > \sqrt{2}$. Then the equation becomes $(x - 1)(x^2 - 2) - 2 = x(x - 2)(x + 1) = 0$. Thus, $x = 2$ is the only rational root for $x > \sqrt{2}$. Consider the case in which $-\sqrt{2} < x < 1$. Then the equation becomes $(x-1)(x^2-2)-2 = x(x-2)(x+1) = 0$. Thus, $x = 0$ and $x = -1$ are the rational roots for $-\sqrt{2} < x < 1$. Consider the case in which $x < -\sqrt{2}$ or the case in which $1 < x < \sqrt{2}$. In these cases, the equation becomes $(1-x)(x^2-2)-2 = -x^3+x^2+2x-4$. By the rational root theorem, the rational roots of this polynomial can only be $\pm 4, \pm 2, \pm 1$ and a quick check shows that none of these are roots, so this polynomial has no rational roots.

5. Let $S = \{1,2,3,4,5,6,7,8,9,10\}$. In how many ways can two (not necessarily distinct) elements a, b be taken from S such that $\frac{a}{b}$ is in lowest terms, i.e. a and b share no common divisors other than 1?

   Answer: 63

   This amounts to determining, for a given numerator, how many elements in S are relatively prime to the numerator. If we let $f(n)$ be the number of positive integers relatively prime to n and less than or equal to 10, it is obvious that $f(1) = 10$, $f(2) = f(4) = f(8) = 5$, $f(3) = f(9) = 7$, $f(5) = 8$, $f(7) = 9$, $f(6) = 3$, and $f(10) = 4$. Therefore, the answer is $10 + 3 \cdot 5 + 2 \cdot 7 + 8 + 9 + 3 + 4 = 63$.

6. Find all square numbers which can be represented in the form $2^a + 3^b$, where a, b are nonnegative integers. You can assume the fact that the equation $3^x - 2^y = 1$ has no integer solutions if $x \geq 3$.

   Answer: $\mathbf{2^2, 3^2, 5^2}$

   For $b = 0$ one has $2^a + 1 = c^2$, $2^a = (c+1)(c-1)$. Thus both $c+1$ and $c-1$ should be powers of 2. The only possibility is $c = 3$, which gives a solution $2^3 + 3^0 = 9 = 3^2$.
   For $b \geq 1$, $2^a + 3^b$ is not divisible by 3, so it should be $\equiv 1 \pmod 3$. This requires a to be even. Let $a = 2d$, then $3^b = c^2 - 2^{2d} = (c + 2^d)(c - 2^d)$. Let $c + 2^d = 3^p$ and $c - 2^d = 3^q$. Eliminating c, one has $2^{d+1} = 3^p - 3^q$. For $q \geq 1$ the right-hand side is divisible by 3, so $q = 0$. From what we know, there are only two solutions $(d, p) = (0, 1), (2, 2)$. These solutions give $2^0 + 3^1 = 4 = 2^2$ and $2^4 + 3^2 = 25 = 5^2$ respectively.

7. A frog is jumping on the number line, starting at zero and jumping to seven. He can jump from x to either $x+1$ or $x+2$. However, the frog is easily confused, and before arriving at the number seven, he will turn around and jump in the wrong direction, jumping from x to $x-1$. This happens exactly once, and will happen in such a way that the frog will not land on a negative number. How many ways can the frog get to the number seven?

   Answer: 146

   Let f_n be the number of ways to jump from zero to n, ignoring for the time being jumping backwards.. We have $f_0 = 1$, $f_1 = 1$, and $f_n = f_{n-1} + f_{n-2}$ when $n \geq 2$. Therefore, we have that $f_2 = 2$, $f_3 = 3$, $f_4 = 5$, $f_5 = 8$, $f_6 = 13$, and $f_7 = 21$. Note that we can describe the frog's jumping as jumping forward n numbers, jumping backward 1 number, and jumping forward $8-n$ numbers. Therefore, the desired answer is simply $\sum_{i=1}^{6} f_i f_{8-i} = \boxed{146}$.

8. Call a nonnegative integer k *sparse* when all pairs of 1's in the binary representation of k are separated by at least two zeroes. For example, $9 = 1001_2$ is sparse, but $10 = 1010_2$ is not sparse. How many sparse numbers are less than 2^{17}?

   Answer: 872

   Let a_n denote the number of sparse numbers with no more than n binary digits. In particular, for numbers with less than n binary digits after removing leading zeroes, append leading zeroes so all numbers have n binary digits when including sufficiently many leading zeroes. We have that $a_0 = 1$, $a_1 = 2$, and $a_2 = 3$ since for these lengths, either zero digits are 1 or one digit is 1. We claim that the recurrence $a = a_{n-1} + a_{n-3}$ holds for $n \geq 3$. We split this analysis into two cases; numbers where the nth binary digit is 0 or 1. When the nth binary digit is zero, we can remove that zero to get a valid number with $n-1$ binary digits. When the nth binary digit is one, it is known that the $(n-1)$th and $(n-2)$th digits are both zero, so we can truncate those to get a valid number with $n-3$ binary digits. Therefore, the recurrence holds. With the given initial conditions, $a_{17} = \boxed{872}$.

9. Two ants begin on opposite corners of a cube. On each move, they can travel along an edge to an adjacent vertex. Find the probability they both return to their starting position after 4 moves.

Answer: $\frac{\mathbf{49}}{\mathbf{729}}$

Let the cube be oriented so that one ant starts at the origin and the other at $(1,1,1)$. Let x, y, z be moves away from the origin and x', y', z' be moves toward the origin in each the respective directions. Any move away from the origin has to at some point be followed by a move back to the origin, and if the ant moves in all three directions, then it can't get back to its original corner in 4 moves. The number of ways to choose 2 directions is $\binom{3}{2} = 3$ and for each pair of directions there are $\frac{4!}{2!2!} = 6$ ways to arrange four moves a, a', b, b' such that a precedes a' and b precedes b'. Hence there are $3 \cdot 6 = 18$ ways to move in two directions. The ant can also move in a, a', a, a' (in other words, make a move, return, repeat the move, return again) in three directions so this gives $18 + 3 = 21$ moves. There are $3^4 = 81$ possible moves, 21 of which return the ant for a probability of $\frac{21}{81} = \frac{7}{27}$. Since this must happen simultaneously to both ants, the probability is $\frac{7}{27} \cdot \frac{7}{27} = \frac{49}{729}$.

10. An unfair coin has a 2/3 probability of landing on heads. If the coin is flipped 50 times, what is the probability that the total number of heads is even?

Answer: $\frac{\mathbf{1+(1/3)^{50}}}{\mathbf{2}}$

The coin can turn up heads 0,2,4,..., or 50 times to satisfy the problem. Hence the probability is

$$P = \binom{50}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{50} + \binom{50}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{48} + \cdots + \binom{50}{50}\left(\frac{2}{3}\right)^{50}\left(\frac{1}{3}\right)^0.$$

Note that this sum is the sum of the even-powered terms of the expansion $(1/3+2/3)^{50}$. To isolate these terms, we note that the odd-powered terms of $(1/3-2/3)^{50}$ are negative. So by adding $(1/3+2/3)^{50} + (1/3-2/3)^{50}$, we get rid of the odd-powered terms and we are left with two times the sum of the even terms. Hence the probability is

$$P = \frac{(1/3+2/3)^{50} + (1/3-2/3)^{50}}{2} = \frac{1+(1/3)^{50}}{2}.$$

11. Find the unique polynomial $P(x)$ with coefficients taken from the set $\{-1, 0, 1\}$ and with least possible degree such that $P(2010) \equiv 1 \pmod 3$, $P(2011) \equiv 0 \pmod 3$, and $P(2012) \equiv 0 \pmod 3$.

Answer: $\boldsymbol{P(x) = 1 - x^2}$

First suppose $P(x)$ is constant or linear. Then we have $P(2010) + P(2012) = 2P(2011)$, which is a contradiction because the left side is congruent to 1 (mod 3) and the right is congruent to 0 (mod 3). So P must be at least quadratic. The space of quadratic polynomials in x is spanned by the polynomials $f(x) = 1$, $g(x) = x$, and $h(x) = x^2$. Applying each of these to 2010, 2011, and 2012, we have the mod 3 equivalences:

$$f(2010, 2011, 2012) \equiv (1,1,1)$$
$$g(2010, 2011, 2012) \equiv (0,1,2)$$
$$h(2010, 2011, 2012) \equiv (0,1,1)$$

Subtracting the third row from the first, we have $P(x) = f(x) - h(x) = 1 - x^2$, giving $P(2010, 2011, 2012) \equiv (1,0,0) \pmod 3$, as desired. Uniqueness follows from the observation that the three vectors above form a basis for $(\mathbb{Z}/3\mathbb{Z})^3$.

12. Let $a, b \in \mathbb{C}$ such that $a + b = a^2 + b^2 = \frac{2\sqrt{3}}{3}i$. Compute $|\operatorname{Re}(a)|$.

Answer: $\frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$

From $a + b = 2\frac{\sqrt{3}}{3}i$ we can let $a = \frac{\sqrt{3}}{3}i + x$ and $b = \frac{\sqrt{3}}{3}i - x$. Then $a^2 + b^2 = 2((\frac{\sqrt{3}}{3}i)^2 + x^2) = 2(x^2 - \frac{1}{3}) = \frac{2\sqrt{3}}{3}i$. So $x^2 = \frac{1+\sqrt{3}i}{3} = \frac{2}{3}e^{i\pi/3}$, $x = \pm\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}+i}{2}$. Since $|\operatorname{Re}(a)| = |\operatorname{Re}(x)|$, the answer is $\frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}}$.

13. Let T_n denote the number of terms in $(x+y+z)^n$ when simplified, i.e. expanded and like terms collected, for non-negative integers $n \geq 0$. Find

$$\sum_{k=0}^{2010}(-1)^k T_k$$
$$= T_0 - T_1 + T_2 - \cdots - T_{2009} + T_{2010}.$$

Answer: $\mathbf{1006^2}$

First note that the expression $(x+y+z)^n$ is equal to

$$\sum \frac{n!}{a!b!c!}x^a y^b z^c$$

where the sum is taken over all non-negative integers a, b, and c with $a+b+c=n$. The number of non-negative integer solutions to $a+b+c=n$ is $\binom{n+2}{2}$, so $T_k = \binom{k+2}{2}$ for $k \geq 0$. It is easy to see that $T_k = 1+2+\cdots+(k+1)$, so T_k is the $(k+1)$st triangular number. If $k = 2n-1$ is odd, then for all positive integers i, $T_{2i} - T_{2i-1} = 2i+1$ and therefore [1]

$$\begin{aligned}\sum_{j=0}^{k-1}(-1)^j T_j &= T_0 + \sum_{j=1}^{n-1}(T_{2j} - T_{2j-1}) \\ &= 1 + \sum_{j=2}^{n}(2j-1) \\ &= n^2.\end{aligned}$$

Therefore, since T_{2010} is the 2011th triangular number and $2011 = 2(1006)-1$, we can conclude that the desired sum is 1006^2.

14. Let $M=(-1,2)$ and $N=(1,4)$ be two points in the plane, and let P be a point moving along the x-axis. When $\angle MPN$ takes on its maximum value, what is the x-coordinate of P?

Answer: 1

Let $P=(a,0)$. Note that $\angle MPN$ is inscribed in the circle defined by points M, P, and N, and that it intercepts MN. Since MN is fixed, it follows that maximizing the measure of $\angle MPN$ is equivalent to minimizing the size of the circle defined by M, P, and N. Since P must be on the x-axis, we therefore want this circle to be tangent to the x-axis. Since the center of this circle must lie on the perpendicular bisector of MN, which is the line $y = 3-x$, the center of the circle has to be of the form $(a, 3-a)$, so a has to satisfy $(a+1)^2 + (1-a)^2 = (a-3)^2$. Solving this equation gives $a=1$ or $a=-7$. Clearly choosing a = 1 gives a smaller circle, so our answer is 1.

15. Consider the curves $x^2+y^2=1$ and $2x^2+2xy+y^2-2x-2y=0$. These curves intersect at two points, one of which is $(1,0)$. Find the other one.

Answer: $\left(-\frac{3}{5}, \frac{4}{5}\right)$

From the first equation, we get that $y^2 = 1-x^2$. Plugging this into the second one, we are left with

$$\begin{aligned}2x^2 \pm 2x\sqrt{1-x^2} + 1 - x^2 - 2x \mp 2\sqrt{1-x^2} = 0 &\Rightarrow (x-1)^2 = \mp 2\sqrt{1-x^2}(x-1) \\ &\Rightarrow x-1 = \mp 2\sqrt{1-x^2} \text{ assuming } x \neq 1 \\ &\Rightarrow x^2-2x+1 = 4-4x^2 \Rightarrow 5x^2-2x-3=0.\end{aligned}$$

The quadratic formula yields that $x = \frac{2\pm 8}{10} = 1, -\frac{3}{5}$ (we said that $x \neq 1$ above but we see that it is still valid). If $x=1$, the first equation forces $y=0$ and we easily see that this solves the second equation. If $x = -\frac{3}{5}$, then clearly y must be positive or else the second equation will sum five positive terms. Therefore $y = \sqrt{1-\frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$. Hence the other point is $\left(-\frac{3}{5}, \frac{4}{5}\right)$.

[1] For a quick visual proof of this fact, we refer the reader to `http://www.jstor.org/stable/2690575`.

16. If r, s, t, and u denote the roots of the polynomial $f(x) = x^4 + 3x^3 + 3x + 2$, find
$$\frac{1}{r^2} + \frac{1}{s^2} + \frac{1}{t^2} + \frac{1}{u^2}.$$

Answer: $\mathbf{\frac{9}{4}}$

First notice that the polynomial
$$g(x) = x^4\left(\frac{1}{x^4} + \frac{3}{x^3} + \frac{3}{x} + 2\right) = 2x^4 + 3x^3 + 3x + 1$$
is a polynomial with roots $\frac{1}{r}$, $\frac{1}{s}$, $\frac{1}{t}$, $\frac{1}{u}$. Therefore, it is sufficient to find the sum of the squares of the roots of $g(x)$, which we will denote as r_1 through r_4. Now, note that
$$r_1^2 + r_2^2 + r_3^2 + r_4^2 = (r_1 + r_2 + r_3 + r_4)^2 - (r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4) = (-\frac{a_3}{a_4})^2 - \frac{a_2}{a_4}$$
by Vieta's Theorem, where a_n denotes the coefficient of x^n in $g(x)$. Plugging in values, we get that our answer is $(-\frac{3}{2})^2 - 0 = \frac{9}{4}$.

17. An *icosahedron* is a regular polyhedron with 12 vertices, 20 faces, and 30 edges. How many rigid rotations G are there for an icosahedron in $\mathbb{R}^3$?

Answer: 60

There are 12 vertices, each with 5 neighbors. Any vertex and any of its neighbors can be rotated to any other vertex-neighbor pair in exactly one way. There are $5 \cdot 12 = 60$ vertex-neighbor pairs.

18. Pentagon $ABCDE$ is inscribed in a circle of radius 1. If $\angle DEA \cong \angle EAB \cong \angle ABC$, $m\angle CAD = 60°$, and $BC = 2DE$, compute the area of $ABCDE$.

Answer: $\mathbf{\frac{33\sqrt{3}}{28}}$

Looking at cyclic quadrilaterals $ABCD$ and $ACDF$ tells us that $m\angle ACD = m\angle ADC$, so $\triangle ACD$ is equilateral and $m\angle DEA = 120°$. Now, if we let $m\angle EAD = \theta$, we see that $m\angle CAB = 60° - \theta \implies m\angle ACB = \theta \implies \triangle AED \cong \triangle CBA$. Now all we have to do is calculate side lengths. After creating some $30° - 60° - 90°$ triangles, it becomes evident that $AC = \sqrt{3}$. Now let $AB = x$, so $BC = 2x$. By applying the Law of Cosines to triangle ABC, we find that $x^2 = \frac{3}{7}$. Hence, the desired area $(ABCDE) = (ACD) + 2(ABC) = \frac{(\sqrt{3})^2\sqrt{3}}{4} + 2 \cdot \frac{1}{2}(x)(2x)(\sin 120°) = \frac{33\sqrt{3}}{28}$.

19. Five students at a meeting remove their name tags and put them in a hat; the five students then each randomly choose one of the name tags from the bag. What is the probability that exactly one person gets their own name tag?

Answer: $\mathbf{\frac{3}{8}}$

Assume without loss of generality that the first person gets a correct nametag. Let's call the other people B, C, D, and E. We can order the four people in nine ways such that none of the persons gets his own nametag; CBED, CDEB, CEBD, DBEC, DEBC, DECB, EBCD, EDBC, EDCB. Therefore, the desired probability is $\frac{9}{4!} = \frac{3}{8}$.

Alternative Solution: The selection of random nametags amounts to a selection of a random permutation of the five students from the symmetric group S_5. The condition will be met if and only if the selected permutation σ has exactly one cycle of length one (i.e., exactly one fixed point). The only distinct cycle types with exactly one fixed point are $(1, 4)$ and $(1, 2, 2)$. There are $\frac{5!}{4} = 30$ permutations of the first type and $\frac{5!}{2^3} = 15$ permutations of the second. Thus, the desired probability is $\frac{30 + 15}{5!} = \frac{3}{8}$.

20. Find the 2011th-smallest x, with $x > 1$, that satisfies the following relation:
$$\sin(\ln x) + 2\cos(3\ln x)\sin(2\ln x) = 0.$$

Answer: $\boldsymbol{x = e^{2011\pi/5}}$

Set $y = \ln x$, and observe that

$$2\cos(3y)\sin(2y) = \sin(3y + 2y) - \sin(3y - 2y) = \sin(5y) - \sin(y),$$

so that the equation in question is simply

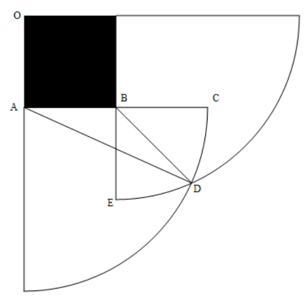
$$\sin(5y) = 0.$$

The solutions are therefore

$$\ln x = y = \frac{n\pi}{5} \implies x = e^{n\pi/5} \quad \text{for all } n \in \mathbb{N}$$

21. An ant is leashed up to the corner of a solid square brick with side length 1 unit. The length of the ant's leash is 6 units, and it can only travel on the ground and not through or on the brick. In terms of $x = \arctan\left(\frac{3}{4}\right)$, what is the area of region accessible to the ant?

    Answer: $\boldsymbol{\frac{79\pi}{2} + \frac{3}{2} - 5x}$



Label the top left corner of the square as the origin O. By keeping the leash straight, the ant can travel through $\frac{3}{4}$ of a circle of radius 6 ($A_1 = \frac{3}{4} \times 36\pi = 27\pi$). The ant can also bend the leash around the two nearest corners of the square to where it is leashed ($A_2 = 2 \times \frac{1}{4} \times 25\pi = \frac{25}{2}\pi$). However, this double counts the area enclosed by BCDE, which is equal to two times the area of BCD. To calculate the latter, notice that ACD is the sector of the circle centered at A with radius 5. We can calculate the coordinates of D usign the two equations $y = -x$ (from the symmetry) and $x^2 + (y+1)^2 = 5^2$ which yields $D = (4, -4)$. Since $A = (0, -1)$, the angle of sector ACD is $\arctan\left(\frac{3}{4}\right) = x$. The area of triangle ABD equals $\frac{3}{2}$ (base times height) so BCD has area $5x - \frac{3}{2}$ and BCDE has area $10x - 3$. Hence, the total area is $A_1 + A_2 - \left(5x - \frac{3}{2}\right) = \frac{79\pi}{2} + \frac{3}{2} - 5x$.

22. Compute the sum of all n for which the equation $2x + 3y = n$ has exactly 2011 nonnegative ($x, y \geq 0$) integer solutions.

    Answer: **72381**

    Observe that if the equation $ax + by = n$ has m solutions, the equation $ax + by = n + ab$ has $m + 1$ solutions. Also note that $ax + by = ax_0 + by_0$ for $0 \leq x_0 < b$, $0 \leq y_0 < a$ has no other solution than $(x, y) = (x_0, y_0)$. (It is easy to prove both if you consider the fact that the general solution has form $(x' + bk, y' - ak)$.) So there are ab such n and their sum is

$$\sum_{\substack{0 \leq x < b \\ 0 \leq y < a}} (ax + by + 2010ab) = 2010a^2b^2 + \frac{ab(2ab - a - b)}{2}.$$

23. Let ABC be any triangle, and D, E, F be points on $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $CD = 2BD$, $AE = 2CE$ and $BF = 2AF$. $\overline{AD}$ and $\overline{BE}$ intersect at X, $\overline{BE}$ and $\overline{CF}$ intersect at Y, and $\overline{CF}$ and $\overline{AD}$ intersect at Z. Find $\frac{Area(\triangle ABC)}{Area(\triangle XYZ)}$.

Answer: 7

Using Menelaus's Theorem on $\triangle ABD$ with collinear points F, X, C and the provided ratios gives $DX/XA = 4/3$. Using Menelaus's Theorem on $\triangle ADC$ with collinear points B, Y, E gives $AY/YD = 6$. We conclude that AX, XY, YD are in length ratio $3 : 3 : 1$. By symmetry, this also applies to the segments CZ, ZX, XF and BY, YZ, ZE. Repeatedly using the fact that the area ratio of two triangles of equal height is the ratio of their bases, we find $[ABC] = (3/2)[ADC] = (3/2)(7/3)[XYC] = (3/2)(7/3)(2)[XYZ] = 7[XYZ]$, or $[ABC]/[XYZ] = 7$.

Alternate Solution

Stretching the triangle will preserve ratios between lengths and ratios between areas, so we may assume that $\triangle ABC$ is equilateral with side length 3. We now use mass points to find the length of XY. Assign a mass of 1 to A. In order to have X be the fulcrum of $\triangle ABC$, C have mass 2 and B must have mass 4. Hence, $BX : XE = 4 : 3$ and $AX : XD = 6 : 1$, the latter of which also equals $BY : YE$ by symmetry. Hence, $XY = \frac{3}{7}BE$. To find BE, we apply the Law of Cosines to $\triangle CBE$ to get that $BE^2 = 1^2 + 3^2 - 2 \cdot 1 \cdot 3 \cdot \cos 60^\circ = 7 \implies XY = \frac{3\sqrt{7}}{7}$. Since $\triangle XYZ$ must be equilateral by symmetry, the desired ratio equals $(\frac{AB}{XY})^2 = 7$.

24. Let $P(x)$ be a polynomial of degree 2011 such that $P(1) = 0$, $P(2) = 1$, $P(4) = 2$, ... , and $P(2^{2011}) = 2011$. Compute the coefficient of the x^1 term in $P(x)$.

Answer: $2 - \frac{1}{2^{2010}}$

We analyze $Q(x) = P(2x) - P(x)$. One can observe that $Q(x) - 1$ has the powers of 2 starting from $1, 2, 4, \cdots$, up to 2^{2010} as roots. Since Q has degree 2011, $Q(x) - 1 = A(x-1)(x-2)\cdots(x-2^{2010})$ for some A. Meanwhile $Q(0) = P(0) - P(0) = 0$, so

$$Q(0) - 1 = -1 = A(-1)(-2)\cdots(-2^{2010}) = -2^{(2010\cdot 2011)/2}A.$$

Therefore $A = 2^{-(1005\cdot 2011)}$. Finally, note that the coefficient of x is same for P and $Q-1$, so it equals $A(-2^0)(-2^1)\cdots(-2^{2010})((-2^0) + (-2^{-1}) + \cdots + (-2^{-2010})) = \frac{A\cdot 2^{1005\cdot 2011}(2^{2011}-1)}{2^{2010}} = \boxed{2 - \frac{1}{2^{2010}}}$.

25. Find the maximum of

$$\frac{ab + bc + cd}{a^2 + b^2 + c^2 + d^2}$$

for reals a, b, c, and d not all zero.

Answer: $\frac{\sqrt{5}+1}{4}$

One has $ab \le \frac{t}{2}a^2 + \frac{1}{2t}b^2$, $bc \le \frac{1}{2}b^2 + \frac{1}{2}c^2$, and $cd \le \frac{1}{2t}c^2 + \frac{t}{2}d^2$ by AM-GM. If we can set t such that $\frac{t}{2} = \frac{1}{2t} + \frac{1}{2}$, it can be proved that $\frac{ab+bc+cd}{a^2+b^2+c^2+d^2} \le \frac{\frac{t}{2}(a^2+b^2+c^2+d^2)}{a^2+b^2+c^2+d^2} = \frac{t}{2}$, and this is maximal because we can set a, b, c, d so that the equality holds in every inequality we used. Solving this equation, we get $t = \frac{1+\sqrt{5}}{2}$, so the maximum is $\frac{t}{2} = \frac{\sqrt{5}+1}{4}$.