1. Five students at a meeting remove their name tags and put them in a hat; the five students then each randomly choose one of the name tags from the bag. What is the probability that exactly one person gets their own name tag?

   **Answer:** $\mathbf{\frac{3}{8}}$

   Assume without loss of generality that the first person gets a correct nametag. Let's call the other people B, C, D, and E. We can order the four people in nine ways such that none of the persons gets his own nametag; CBED, CDEB, CEBD, DBEC, DEBC, DECB, EBCD, EDBC, EDCB. Therefore, the desired probability is $\frac{9}{4!} = \frac{3}{8}$.

   **Alternative Solution:** The selection of random nametags amounts to a selection of a random permutation of the five students from the symmetric group $S_5$. The condition will be met if and only if the selected permutation $\sigma$ has exactly one cycle of length one (i.e., exactly one fixed point). The only distinct cycle types with exactly one fixed point are $(1,4)$ and $(1,2,2)$. There are $\dfrac{5!}{4} = 30$ permutations of the first type and $\dfrac{5!}{2^3} = 15$ permutations of the second. Thus, the desired probability is $\dfrac{30+15}{5!} = \dfrac{3}{8}$.

2. Compute

   $$\sum_{n=1}^{\infty} \frac{(7n+32)\cdot 3^n}{n\cdot(n+2)\cdot 4^n}.$$

   **Answer:** $\mathbf{\frac{33}{2}}$

   Note that $\frac{7n+32}{n(n+2)} = \frac{16}{n} - \frac{9}{n+2}$ so that

   $$\begin{aligned}\sum_{n=1}^{\infty} \frac{(7n+32)}{n(n+2)}\frac{3^n}{4^n} &= \sum_{n=1}^{\infty}\frac{16}{n}\frac{3^n}{4^n} - \sum_{n=1}^{\infty}\frac{9}{n+2}\frac{3^n}{4^n}\\ &= \sum_{n=1}^{\infty}\frac{16}{n}\frac{3^n}{4^n} - \sum_{n=1}^{\infty}\frac{16}{n+2}\frac{3^{n+2}}{4^{n+2}}\\ &= \sum_{n=1}^{\infty}\frac{16}{n}\frac{3^n}{4^n} - \sum_{n=3}^{\infty}\frac{16}{n}\frac{3^n}{4^n}\\ &= \frac{16}{1}\frac{3}{4} + \frac{16}{2}\frac{9}{16} = \frac{33}{2}.\end{aligned}$$

3. Find the unique polynomial $P(x)$ with coefficients taken from the set $\{-1,0,1\}$ and with least possible degree such that $P(2010) \equiv 1 \pmod 3$, $P(2011) \equiv 0 \pmod 3$, and $P(2012) \equiv 0 \pmod 3$.

   **Answer:** $\boldsymbol{P(x) = 1 - x^2}$

   First suppose $P(x)$ is constant or linear. Then we have $P(2010) + P(2012) = 2P(2011)$, which is a contradiction because the left side is congruent to 1 (mod 3) and the right is congruent to 0 (mod 3). So $P$ must be at least quadratic. The space of quadratic polynomials in $x$ is spanned by the polynomials $f(x) = 1$, $g(x) = x$, and $h(x) = x^2$. Applying each of these to 2010, 2011, and 2012, we have the mod 3 equivalences:

   $$f(2010, 2011, 2012) \equiv (1,1,1)$$
   $$g(2010, 2011, 2012) \equiv (0,1,2)$$
   $$h(2010, 2011, 2012) \equiv (0,1,1)$$

   Subtracting the third row from the first, we have $P(x) = f(x) - h(x) = 1 - x^2$, giving $P(2010, 2011, 2012) \equiv (1,0,0) \pmod 3$, as desired. Uniqueness follows from the observation that the three vectors above form a basis for $(\mathbb{Z}/3\mathbb{Z})^3$.

4. Let $T_n$ denote the number of terms in $(x+y+z)^n$ when simplified, i.e. expanded and like terms collected, for non-negative integers $n \geq 0$. Find

$$\sum_{k=0}^{2010}(-1)^k T_k = T_0 - T_1 + T_2 - \cdots - T_{2009} + T_{2010}.$$

**Answer: $\mathbf{1006^2}$**

First note that the expression $(x+y+z)^n$ is equal to

$$\sum \frac{n!}{a!b!c!} x^a y^b z^c$$

where the sum is taken over all non-negative integers $a$, $b$, and $c$ with $a+b+c=n$. The number of non-negative integer solutions to $a+b+c=n$ is $\binom{n+2}{2}$, so $T_k = \binom{k+2}{2}$ for $k \geq 0$. It is easy to see that $T_k = 1+2+\cdots+(k+1)$, so $T_k$ is the $(k+1)$st triangular number. If $k = 2n-1$ is odd, then for all positive integers $i$, $T_{2i} - T_{2i-1} = 2i+1$ and therefore [1]

$$\begin{aligned} \sum_{j=0}^{k-1}(-1)^j T_j &= T_0 + \sum_{j=1}^{n-1}(T_{2j} - T_{2j-1}) \\ &= 1 + \sum_{j=2}^{n}(2j-1) \\ &= n^2. \end{aligned}$$

Therefore, since $T_{2010}$ is the 2011th triangular number and $2011 = 2(1006) - 1$, we can conclude that the desired sum is $1006^2$.

5. Two ants begin on opposite corners of a cube. On each move, they can travel along an edge to an adjacent vertex. Find the probability they both return to their starting position after 4 moves.

**Answer: $\mathbf{\frac{49}{729}}$**

Let the cube be oriented so that one ant starts at the origin and the other at $(1,1,1)$. Let $x, y, z$ be moves away from the origin and $x', y', z'$ be moves toward the origin in each the respective directions. Any move away from the origin has to at some point be followed by a move back to the origin, and if the ant moves in all three directions, then it can't get back to its original corner in 4 moves. The number of ways to choose 2 directions is $\binom{3}{2} = 3$ and for each pair of directions there are $\frac{4!}{2!2!} = 6$ ways to arrange four moves $a, a', b, b'$ such that $a$ precedes $a'$ and $b$ precedes $b'$. Hence there are $3 \cdot 6 = 18$ ways to move in two directions. The ant can also move in $a, a', a, a'$ (in other words, make a move, return, repeat the move, return again) in three directions so this gives $18 + 3 = 21$ moves. There are $3^4 = 81$ possible moves, 21 of which return the ant for a probability of $\frac{21}{81} = \frac{7}{27}$. Since this must happen simultaneously to both ants, the probability is $\frac{7}{27} \cdot \frac{7}{27} = \frac{49}{729}$.

6. An unfair coin has a 2/3 probability of landing on heads. If the coin is flipped 50 times, what is the probability that the total number of heads is even?

**Answer: $\mathbf{\frac{1+(1/3)^{50}}{2}}$**

The coin can turn up heads 0,2,4,..., or 50 times to satisfy the problem. Hence the probability is

$$P = \binom{50}{0}\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^{50} + \binom{50}{2}\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^{48} + \cdots + \binom{50}{50}\left(\frac{2}{3}\right)^{50}\left(\frac{1}{3}\right)^0.$$

Note that this sum is the sum of the even-powered terms of the expansion $(1/3 + 2/3)^{50}$. To isolate these terms, we note that the odd-powered terms of $(1/3 - 2/3)^{50}$ are negative. So by adding $(1/3 +$

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[1] For a quick visual proof of this fact, we refer the reader to `http://www.jstor.org/stable/2690575`.

$2/3)^{50} + (1/3 - 2/3)^{50}$, we get rid of the odd-powered terms and we are left with two times the sum of the even terms. Hence the probability is

$$P = \frac{(1/3 + 2/3)^{50} + (1/3 - 2/3)^{50}}{2} = \frac{1 + (1/3)^{50}}{2}.$$

7. Compute the sum of all $n$ for which the equation $2x + 3y = n$ has exactly 2011 nonnegative $(x, y \geq 0)$ integer solutions.

   **Answer: 72381**

   Observe that if the equation $ax + by = n$ has $m$ solutions, the equation $ax + by = n + ab$ has $m + 1$ solutions. Also note that $ax + by = ax_0 + by_0$ for $0 \leq x_0 < b$, $0 \leq y_0 < a$ has no other solution than $(x, y) = (x_0, y_0)$. (It is easy to prove both if you consider the fact that the general solution has form $(x' + bk, y' - ak)$.) So there are $ab$ such $n$ and their sum is

$$\sum_{\substack{0 \leq x < b \\ 0 \leq y < a}} (ax + by + 2010ab) = 2010a^2b^2 + \frac{ab(2ab - a - b)}{2}.$$

8. Let $\{a_i\}_{i=1,2,3,4}, \{b_i\}_{i=1,2,3,4}, \{c_i\}_{i=1,2,3,4}$ be permutations of $\{1, 2, 3, 4\}$. Find the minimum of $a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 + a_4b_4c_4$.

   **Answer: 44**

   The minimum can be obtained by

$$1 \cdot 3 \cdot 4 + 2 \cdot 2 \cdot 3 + 3 \cdot 4 \cdot 1 + 4 \cdot 1 \cdot 2 = 12 + 12 + 12 + 8 = 44.$$

   We claim that 44 is optimum. Denote $x_i = a_ib_ic_i$. Since $x_1x_2x_3x_4 = (1 \cdot 2 \cdot 3 \cdot 4)^3 = 2^9 \cdot 3^3$, $x_i$ should only consist of prime factors of 2 and 3. So between 8 and 12 $x_i$ can only be 9.

   Case 1. There are no 9 among $x_i$ Then $x_i$ are not in $(8, 12)$. And $x_1x_2x_3x_4 = 12 \cdot 12 \cdot 12 \cdot 8$, so if $x_1$ is minimum then $x_1 \leq 8$. Then by AM-GM inequality $x_2 + x_3 + x_4 \geq 3(x_2x_3x_4)^{1/3}$. If we let $(x_2x_3x_4)^{1/3} = 12y$ then $x_1 = 8y^{-3}$, and for $y \geq 1$ $8y^{-3} + 36y$ attains minimum at $y = 1$. So $x_1 + x_2 + x_3 + x_4 \geq 8y^{-3} + 36y \geq 44$.

   Case 2. $x_1$ is 9. Then $x_2x_3x_4$ is divisible by 3 but not 9. So only $x_2$ is divisible by 3 and others are just powers of 2. $x_2$ can be 3, 6, 12, 24 or larger than 44.

   Case 2-1 $x_2 = 3 :$ $x_3x_4 = 2^9$, $x_3 + x_4 \geq 2^5 + 2^4 = 48 > 44$.
   Case 2-2 $x_2 = 6 :$ $x_3x_4 = 2^8$, $x_3 + x_4 \geq 2^4 + 2^4 = 32$, $x_1 + x_2 + x_3 + x_4 \geq 9 + 6 + 32 = 47$.
   Case 2-3 $x_2 = 12 :$ $x_3x_4 = 2^7$, $x_3 + x_4 \geq 2^4 + 2^3 = 24$, $x_1 + x_2 + x_3 + x_4 \geq 9 + 12 + 24 = 45$.
   Case 2-4 $x_2 = 24 :$ $x_3x_4 = 2^6$, $x_3 + x_4 \geq 2^3 + 2^3 = 16$, $x_1 + x_2 + x_3 + x_4 \geq 9 + 24 + 16 = 49$.

9. How many functions $f$ that take $\{1, 2, 3, 4, 5\}$ to $\{1, 2, 3, 4, 5\}$, not necessarily injective or surjective (i.e. one-to-one or onto), satisfy $f(f(f(x))) = f(f(x))$ for all $x$ in $\{1, 2, 3, 4, 5\}$?

   **Answer: 756**

   For any such function $f$, let $A = \{n \mid f(n) = n\}$ be the set of elements fixed by $f$ and let $B = \{n \mid f(n) \in A \text{ and } n \notin A\}$ be the set of elements that are sent to an element in $A$, but are not themselves in $A$. Finally, let $C = \{1, 2, 3, 4, 5\} \setminus (A \cup B)$ be everything else. Note that any possible value of $f(f(x))$ is in $A$ so $A$ is not empty. We will now proceed by considering all possible sizes of $A$.

   (a) $A$ has one element: Without loss of generality, let $f(1) = 1$, so we will multiply our result by 5 at the end to account for the other possible values. Suppose that $B$ has $n$ elements so $C$ has the remaining $4 - n$ elements. Since $f(f(x)) = 1$ for each $x$ so any element $c$ in $C$ must satisfy $f(c) = b$ for some $b$ in $B$, because $f(c) \neq 1$ and the only other numbers for which $f(x) = 1$ are the elements of $B$. This also implies that $B$ is not empty. Conversely, any function satisfying

$f(c) = b$ works, so the total number of functions in this case is $5\sum_{n=1}^{4}\binom{4}{n}n^{4-n}$ because there are $\binom{4}{n}$ ways to choose the elements in $B$, and each of the $4-n$ elements in $C$ can be sent to any element of $B$ (there are $n$ of them). This sum is equal to $5(4+6\cdot 4+4\cdot 3+1) = 205$, so there are 205 functions in this case that $A$ has one element.

(b) $A$ has two elements: This is similar to the first case, except that each element in $B$ can now correspond to one of two possible elements in $A$, so this adds a factor of $2^n$. The sum now becomes $\binom{5}{2}\sum_{n=1}^{3}\binom{3}{n}2^n n^{3-n} = 10(3\cdot 2+3\cdot 4\cdot 2+8) = 380$, so there are 380 functions in this case.

(c) $A$ has three elements: This is again similar to the prior cases, except there are 3 possible targes in $A$, adding a factor of $3^n$. Then the sum is $\binom{5}{3}\sum_{n=1}^{2}\binom{2}{n}3^n n^{2-n} = 10(2\cdot 3+9) = 150$, so there are 150 functions in this case.

(d) $A$ has four elements: The logic is the same as the prior cases and there are $5(4) = 20$ functions in this case.

(e) $A$ has five elements: The identity function is the only possible function in this case.

Adding together the five cases, we see that there are $205+380+150+20+1 = 756$ such functions.

10. Find the number of ways of filling a $2\times 2\times 8$ box with 16 $1\times 1\times 2$ boxes (rotations and reflections of the $2\times 2\times 8$ box are considered distinct).

**Answer: 23409**

Let $a_n$ be the number of ways of filling the $2\times 2\times n$ box, and let $b_n$ be the number of ways of filling it with one $1\times 1\times 2$ box fixed at the "bottom face" ($2\times 2$ face). It is easy to see that $b_n = a_{n-1} + b_{n-1}$. It is then simple to verify that $a_n = 2b_n + 2b_{n-1} + a_{n-2}$. The base cases $a_1 = 2$, $b_1 = 1$, $a_2 = 9$, and $b_2 = 3$ are trivial to calculate. Using these values to calculate $a_8$ recursively gives $a_8 = 23409$.