1. Compute the slope of the line tangent to $y^2 = x^3 + x + 1$ at the point $(0, 1)$.

   Answer: $\frac{1}{2}$

   Solution: We can use implicit differentiation to get $2y \cdot \frac{dy}{dx} = 3x^2 + 1$. Plugging in the point $(0, 1)$, we find $2 \cdot 1 \cdot \frac{dy}{dx} = 3 \cdot 0^2 + 1$ and so $\frac{dy}{dx} = \boxed{\frac{1}{2}}$.

2. For how many real numbers x do we have that $\log_5(1 + x) = x$?

   Answer: 2

   Solution: We begin by noting that there can be no solutions with $x \leq -1$, as the left side is not defined in this region. Furthermore, $x = 0$ is a solution. For any $x > 0$, rewrite the equation as $1 + x = 5^x$. Since $1 + x \leq e^x \leq 5^x$ (which can be shown with, say, taking derivatives or power series) there can be no solutions with $x > 0$.

   So, the only remaining interval we have to check is $-1 < x < 0$. Let $f(x) = 5^x - x - 1$. Then, $f(0) = 0$ and $f(-1) = \frac{1}{5} > 0$. However, $f'(x) = \ln 5 \cdot 5^x - 1$, which satisfies $f'(x) > 0$ whenever $x > -\log_5 \ln 5$. This implies that, for example, $f(-\log_5 \ln 5) < 0$. However, by the Intermediate Value Theorem, on $-\log_5 \ln 5$ and -1, there must be some x between them such that $f(x) = 0$. In fact, this x must also be unique: $f'(x) < 0$ for all $-1 \leq x < -\log_5 \ln 5$. Therefore, our answer is $\boxed{2}$.

3. Eric is standing on the circumference of a circular barn with a radius of 100 meters. Ross starts at the point on the circumference diametrically opposite to Eric and starts moving toward him along the circumference such that the straight-line distance between them decreases at a constant rate of 1 meter per second. When Ross is at an angle of $\frac{\pi}{2}$ from Eric, what is the rate of change of the angle between them, in radians per second? (The angle between Ross and Eric is measured with respect to the center of the circular barn.)

   Answer: $-\frac{\sqrt{2}}{100}$

   Solution: Let $s(t)$ be the straight-line distance between Ross and Eric at time t. Then, we are given that $\frac{ds}{dt} = -1$. Through the Law of Cosines, we have that $\cos(\theta(t)) = \frac{2 \cdot 100^2 - s^2}{2 \cdot 100^2}$. Then, taking a derivative with respect to t it follows that

   $$-\frac{d\theta}{dt} \cdot \sin(\theta) = -\frac{s \cdot \frac{ds}{dt}}{100^2}.$$

   Substituting $s = 100\sqrt{2}$ and $\frac{ds}{dt} = -1$, it follows that $\frac{d\theta}{dt} = \boxed{-\frac{\sqrt{2}}{100}}$.

4. The function $f(x, y)$ has value $-\ln(a)$ whenever $x^2 + \frac{y^2}{4} = a^2$ and $0 < a \leq 1$, and 0 otherwise. Compute the volume contained in the region below this function and above the xy-plane.

   Answer: π

   Solution 1: Let V be the desired volume. We begin by transforming the given ellipse to a circle: that is, set $f(x, y)$ to be $-\ln a$ when $x^2 + y^2 = a^2$. If V' is the new volume, note that $V = 2V'$ (think of this as doing a u-sub of $y = 2z$).

   Now, decompose the volume as the sum of heights along each circular circumference. In essence, unfold the circle $x^2 + y^2 = a^2$ into a line of length $2\pi a$ with value $-\ln a$. We can now directly compute this integral as

$$V = 2\int_0^1 -2\pi a \ln a \, \mathrm{d}a = 4\pi \int_0^1 -a \ln a \, \mathrm{d}a = 4\pi \left(\frac{a^2}{4} - \frac{a^2 \ln a}{2} \Big|_{a=0}^{1} \right) = \boxed{\pi}.$$

Solution 2: Once more do the transformation above to make the ellipse into a circle. Then, the resulting volume is exactly the volume obtained by rotating $f(x) = -\ln x$ about the y axis. Rewriting as $x = e^{-y}$, it follows that

$$V = 2\int_0^\infty \pi (e^{-y})^2 \, \mathrm{d}y = 2\pi \int_0^\infty e^{-2y} \, \mathrm{d}y = \boxed{\pi}.$$

Solution 3: Since $f(x, y) \geq 0$, we are just looking for $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, \mathrm{d}y \, \mathrm{d}x$.

Consider the u-substitution $y = 2u$, so $\mathrm{d}y = 2\mathrm{d}u$. Then,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, \mathrm{d}y \, \mathrm{d}x = 2\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, 2u) \, \mathrm{d}u \, \mathrm{d}x$$

Note that $f(x, 2u)$ has value $-\ln a$ whenever $x^2 + u^2 = a^2$ (this is a formalization of the first solution's approach, in a way). Hence, we can reparametrize in polar coordinates (a, θ) so that

$$\begin{aligned} 2\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, 2u) \, \mathrm{d}u \, \mathrm{d}x &= 2\int_0^1 \int_0^{2\pi} -a \ln a \, \mathrm{d}\theta \, \mathrm{d}a \\ &= 4\pi \int_0^1 -a \ln a \, \mathrm{d}a \\ &= 4\pi \left(\frac{a^2}{4} - \frac{a^2 \ln a}{2} \Big|_{a=0}^{1} \right) \\ &= \boxed{\pi}. \end{aligned}$$

5. Compute

$$\int \sin(\sin(x)) \sin(2x) \, \mathrm{d}x.$$

Answer: $\mathbf{2(\sin(\sin(x)) - \sin(x)\cos(\sin(x)))}$

Solution: Rewrite $\sin(2x) = 2 \sin x \cos x$. Then, if $f(x) = \sin x$, we are looking at

$$2\int f(x) f(f(x)) f'(x) \, \mathrm{d}x$$

. This suggests letting $y = f(x)$, so then our integral becomes

$$2\int y f(y) \, \mathrm{d}y = 2\int \sin(y) \, \mathrm{d}y = 2(\sin(y) - y\cos(y)) = \boxed{2(\sin(\sin(x)) - \sin(x)\cos(\sin(x)))}.$$

6. Compute

$$\int_{-1}^{1} \frac{1 + \cos(x)}{1 + 3^x} \, \mathrm{d}x.$$

Answer: $\mathbf{1 + \sin(1)}$

Solution: Note that

$$\begin{aligned}\int_{-1}^{1} \frac{1+\cos(x)}{1+3^x}\,\mathrm{d}x &= \int_{-1}^{0} \frac{1+\cos(x)}{1+3^x}\,\mathrm{d}x + \int_{0}^{1} \frac{1+\cos(x)}{1+3^x}\,\mathrm{d}x \\ &= -\int_{1}^{0} \frac{1+\cos(-y)}{1+3^{-y}}\,\mathrm{d}y + \int_{0}^{1} \frac{1+\cos(x)}{1+3^x}\,\mathrm{d}x \\ &= \int_{0}^{1} \frac{3^x(1+\cos(x))}{1+3^x} + \frac{1+\cos(x)}{1+3^x}\,\mathrm{d}x \\ &= \int_{0}^{1} 1+\cos(x)\,\mathrm{d}x \\ &= \boxed{1+\sin(1)}\end{aligned}$$

where in the second step we did the u-substitution $y = -x$.

7. A stone is bouncing on a pond. It starts at height 1. Each time it bounces on the pond, its height x changes to a uniformly random height between 0 and x. If the height ever drops below $\frac{1}{10}$, the next time it hits the pond it will sink. What is the expected number of times the stone will bounce before sinking (not counting the sinking as a bounce)?

   Answer: $\mathbf{1 + \ln 10}$

   Solution: Let $f(x)$ be the expected number of times the stone bounces before sinking, starting from height x. Then, $f(x) = 1 + \frac{1}{x}\int_{1/10}^{x} f(y)\,\mathrm{d}y$ if $x \geq \frac{1}{10}$ by conditioning on the next height. Multiplying through by x and taking a derivative yields that $f(x) + xf'(x) = 1 + f(x)$, so $f'(x) = \frac{1}{x}$ and $f(x) = \ln x + C$. The boundary condition is $\lim_{x\to\frac{1}{10}^+} f(x) = 1$, so $C = \ln 10 + 1$. Hence, $f(1) = 1 + \ln 1 + \ln 10 = \boxed{1+\ln 10}$.

8. Let $r_1(t) \leq r_2(t) \leq r_3(t)$ be the roots of $x^3 + tx + 2$. When $t = -3$, compute $r_1'(t)$.

   Answer: $\mathbf{\frac{2}{9}}$

   Solution: Note that when $t = -3$, the polynomial factors as $(x-1)^2(x+2)$, hence $r_1(-3) = -2$. Let's find $r_1'(t)$ using the limit definition of the derivative. In particular, $r_1'(t) = \lim_{h\to 0} \frac{r_1(t+h)-r_1(t)}{h}$. Suppose that $r_1(t+h) = -2+\varepsilon$. Then, $r_1'(t) = \lim_{h\to 0}\frac{\varepsilon}{h}$, where ε is a function of h.

   Now, we have that $(-2+\varepsilon)^3 + (-3+h)(-2+\varepsilon) + 2 = 0$ as $-2+\varepsilon$ is a root of $x^3 + (-3+h)x + 2$ Expanding and cancelling $(-2)^3 + (-3)(-2) + 2 = 0$, we have $12\varepsilon - 6\varepsilon^2 + \varepsilon^3 - 3\varepsilon - 2h + \varepsilon h = 0$. Combining like terms and dividing by h, it follows that

   $$\frac{\varepsilon}{h} = \frac{2}{9} + \frac{2\epsilon^2}{3h} - \frac{\varepsilon^3}{9h} - \varepsilon$$

   By the problem, we know that $\lim_{h\to 0}\frac{\varepsilon}{h} = c$ exists (this can be proven by the continuity of polynomial roots, using the Implicit Function Theorem). In particular, this implies that $\lim_{h\to 0}\varepsilon = 0, \lim_{h\to 0}\frac{\varepsilon^2}{h} = 0, \lim_{h\to 0}\frac{\varepsilon^3}{h} = 0$. Hence, $\lim_{h\to 0}\frac{\varepsilon}{h} = \boxed{\frac{2}{9}}$.

9. Let $\triangle ABC$ be an equilateral triangle with side length 4. Let P be a point chosen uniformly and at random in the interior of $\triangle ABC$. Determine the probability that a square of side length 1 with a corner at P can be rotated to lie entirely within $\triangle ABC$.

   Answer: $\mathbf{1 - \frac{\pi}{48}}$

Solution: Let's look at the bottom left corner. In particular, when is it possible to create such a square cornered at $(p,q) = (r\cos\theta, r\sin\theta)$ for a given angle θ (that is, what is the smallest possible value of r such that we can create this square)

The edges of the triangle in this bottom left corner at $y = 0$ and $y = \sqrt{3}x$, assuming we place the corner at $(0,0)$. Then, the two lines forming the square and emanating from (p,q) are $y = mx + (q - pm)$ and $y = -\frac{1}{m}x + (q + \frac{p}{m})$, parametrized by the slope m. We want the distance from (p,q) to each of the triangle edges to be at least 1.

The squared distance from the first line to $y = 0$ is $(p - \frac{q-pm}{m})^2 + q^2 = q^2\left(1 + \frac{1}{m^2}\right)$.

To find the squared distance from the second line to $y = \sqrt{3}x$, first note that the two intersect at $x = \frac{qm+p}{m\sqrt{3}+1}$ and $y = \sqrt{3}\frac{qm+p}{m\sqrt{3}+1}$. So, the squared distance between the two is $\left(p - \frac{qm+p}{m\sqrt{3}+1}\right)^2 + \left(q - \sqrt{3}\frac{qm+p}{m\sqrt{3}+1}\right)^2 = \left(\frac{q-p\sqrt{3}}{1+m\sqrt{3}}\right)^2(1+m^2)$.

Note that for a given (p,q) the maximum square we can fit into this corner will happen when the two above distances are equal, as the side length of the square is the minimum of the two distances.

So, we can rewrite to have

$$q^2\left(1 + \frac{1}{m^2}\right) = \left(\frac{q - p\sqrt{3}}{1 + m\sqrt{3}}\right)^2(1 + m^2) \iff 3m^2 + 2m\sqrt{3} + 1 = m^2(1 - \cot(\theta)\sqrt{3})^2.$$

Letting $t = \cot(\theta)\sqrt{3} - 1$ and noting that $t > 0$ when $0 \le \theta \le \frac{\pi}{3}$, it follows that the solutions are

$$m = \frac{-2\sqrt{3} \pm \sqrt{12 - 4(3 - t^2)}}{2(3 - t^2)} = \frac{1}{-\sqrt{3} \mp t}.$$

In fact, we can uniquely take the branch $m = \frac{1}{-\sqrt{3}-t}$ because this corresponds to m being negative always (as $\theta \to 0^+$, $m \to 0^-$). Plugging this in to the squared distance from the first line to $y = 0$, we have that the squared distance is

$$d^2 = r^2\sin^2(\theta)(1 + (-\sqrt{3} - (\cot(\theta)\sqrt{3} - 1))^2) = r^2\sin^2(\theta)(1 + (1 - \sqrt{3}(1 + \cot(\theta)))^2).$$

The "forbidden region" of points P corresponding to this corner is then when $d^2 < 1$, or

$$r(\theta) < \frac{1}{\sqrt{\sin^2(\theta)(1 + (1 - \sqrt{3}(1 + \cot(\theta)))^2)}}$$

Therefore, the area of this forbidden region is

$$I = \int_0^{\frac{\pi}{3}}\int_0^{r(\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{3}} \frac{1}{\sin^2(\theta)(1 + (1 - \sqrt{3}(1 + \cot(\theta)))^2)}\,\mathrm{d}\theta.$$

Our final answer will be $1 - \frac{3I}{\frac{4^2\sqrt{3}}{4}} = 1 - \frac{3I}{4\sqrt{3}}$, as none of these three corner regions overlap (due to the square sidelength chosen). So, it suffices to compute the given integral.

Take $x = 1 - \sqrt{3}(1 + \cot(\theta))$, so that $\mathrm{d}x = \sqrt{3}\csc^2\theta\,\mathrm{d}\theta$. Then,

$$I = \frac{1}{2\sqrt{3}}\int_{-\infty}^{-\sqrt{3}} \frac{1}{1 + x^2}\,\mathrm{d}x = \frac{1}{2\sqrt{3}}\arctan(x)\Big|_{x=-\infty}^{-\sqrt{3}} = \frac{1}{2\sqrt{3}}\left(-\frac{\pi}{3} + \frac{\pi}{2}\right) = \frac{\pi}{12\sqrt{3}}.$$

Hence, our answer is $\boxed{1 - \frac{\pi}{48}}$.

10. Define the double factorial via $(2n-1)!! = (2n-1)(2n-3)\cdots 1$. Compute the unique pair (a, c) with $c > 0$ and $a \in (0, \infty)$ such that

$$\lim_{n\to\infty} \frac{c^n(4n-1)!!}{(2n-1)!!(2n-1)!!} = a.$$

Answer: $\left(\frac{\sqrt{2}}{2}, \frac{1}{4}\right)$

Solution 1: We claim that $(a, c) = \boxed{\left(\frac{\sqrt{2}}{2}, \frac{1}{4}\right)}$ is the answer.

First, rewrite

$$P_n = \frac{(4n-1)!!}{(2n-1)!!(2n-1)!!} = \prod_{k=1}^{n} \frac{(4k-1)(4k-3)}{(2k-1)^2}$$

and as $(4k-1)(4k-3) \le 4(2k-1)^2 = (4k-2)(4k-2)$, it follows that $P_n \le 4^n$ so if $c < \frac{1}{4}$ the value of this limit would be 0.

From the other end, we claim that $P_n \ge \left(\frac{1}{2} + \frac{1}{4n}\right)4^n$, implying that indeed $c = \frac{1}{4}$.

To do so, we proceed by induction. Note that $P_1 = 3$ which satisfies the hypothesis. Now, note that

$$P_{n+1} = P_n \cdot \frac{(4n+3)(4n+1)}{(2n+1)^2} = 4P_n \cdot \left(1 - \frac{1}{(4n+2)^2}\right) \ge \left(\frac{1}{2} + \frac{1}{4n}\right)\left(1 - \frac{1}{(4n+2)^2}\right)4^{n+1}$$

and

$$\begin{aligned}\left(\frac{1}{2} + \frac{1}{4n}\right)\left(1 - \frac{1}{(4n+2)^2}\right) &= \frac{1}{2} + \frac{1}{4n} - \frac{1}{2(4n+2)^2} - \frac{1}{4n(4n+2)^2} \\ &= \frac{1}{2} + \frac{3}{16n} + \frac{1}{16n+8} \\ &\ge \frac{1}{2} + \frac{4}{16n+16} \\ &= \frac{1}{2} + \frac{1}{4(n+1)}\end{aligned}$$

so our induction is complete.

Finally, we show that $Q_n = P_n 4^{-n} \to \frac{\sqrt{2}}{2}$. To do so, consider writing

$$Q(z) = \prod_{n=0}^{\infty}\left(1 - \frac{z^2}{\pi^2(n+\frac{1}{2})^2}\right) = \prod_{n=0}^{\infty}\left(1 - \frac{z}{\pi(n+\frac{1}{2})}\right)\left(1 + \frac{z}{\pi(n+\frac{1}{2})}\right)$$

and note that our desired answer is $Q(\frac{\pi}{4})$.

Our (surprising) claim is that in fact $Q(z) = \cos(z)$: writing $\cos(z)$ as a Taylor series gives that it is a polynomial with first coefficient 1, and the zeros of $Q(z)$ are exactly those of $\cos(z)$ (with the same multiplicities, as $\cos(z)$ and $\cos(z)' = \sin(z)$ share no zeros). To show formal convergence, we appeal to the Weierstrass Factorization Theorem, which guarantees such a representation (maybe insert a more formal convergence statement).

Now, we have $Q(\frac{\pi}{4}) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$ and we are done.

Solution 2: Note that $(4n-1)!! = \frac{(4n)!}{(4n)!!} = \frac{(4n)!}{(2n)!\cdot 2^{2n}}$ and similarly $(2n-1)!! = \frac{(2n)!}{n!\cdot 2^n}$. So, we can rewrite

$$\frac{(4n-1)!!}{(2n-1)!!(2n-1)!!} = \frac{\binom{4n}{2n}}{\binom{2n}{n}}.$$

Define $f(n) \sim g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. Then, we claim that

$$\binom{2n}{n} \sim \frac{2^{2n}}{\sqrt{\pi n}}.$$

Indeed, by Stirling's Approximation,

$$\binom{2n}{n} \sim \frac{\sqrt{4\pi n}\left(\frac{2n}{e}\right)^{2n}}{\left(\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\right)^{2}} = \frac{2^{2n}}{\sqrt{\pi n}}.$$

Hence,

$$\frac{\binom{4n}{2n}}{\binom{2n}{n}} \sim \frac{\frac{2^{4n}}{\sqrt{2\pi n}}}{\frac{2^{2n}}{\sqrt{\pi n}}} = \frac{4^n}{\sqrt{2}}.$$

This immediately implies $(a, c) = \boxed{\left(\frac{\sqrt{2}}{2}, \frac{1}{4}\right)}$.