This is set 1.

1. Let A and B be points $(0, 9)$ and $(16, 3)$ respectively on a Cartesian plane. Let point C be the point $(a, 0)$ on the x-axis such that $AC + CB$ is minimized. What is the value of a?

2. William is popping 2022 balloons to celebrate the new year. For each popping round he has two attacks that have the following effects:

   (a) halve the number of balloons (William can not halve an odd number of balloons)

   (b) pop 1 balloon

   How many popping rounds will it take for him to finish off all the balloons in the least amount of moves?

3. What is the numerical value of $(\log_{s^2} m^5)(\log_{m^3} t^6)(\log_{t^5} s^8)$?

This is set 2.

4. Arpit is hanging Christmas lights on his Christmas tree for the holiday season. He decides to hang 12 rows of lights, but if any row of lights is defective then the Christmas tree will not be lit. If the tree is not lit when he plugs in his lights, how many subsets of rows of lights can be broken for the lights to not work?

5. Let $(1 + 2x + 4x^2)^{2020} = a_0 + a_1x + ... + a_{4040}x^{4040}$. Compute the largest exponent k such that 2^k divides
$$\sum_{n=1}^{2020} a_{2n-1}.$$

6. Frank is trying to sort his online friends into groups of sizes n and $n + 2$, for some unknown positive integer n, such that each friend is placed into exactly one group; there can be any number of groups of each of the two sizes. He finds that it is impossible to do so with his current number of friends, but would be possible if he had any *even* number of additional friends. If Frank has less than 400 friends, what is the maximum possible number of friends he has currently?

This is set 3.

7. How many 9-digit numbers are there with unique digits from 1 to 9 such that the first five digits form an increasing series and the last five digits form a decreasing series?

8. Compute the number of ordered triples (a, b, c) with $0 \leq a, b, c \leq 30$ such that 73 divides $8^a + 8^b + 8^c$.

9. Mark plays a game with a circle that has six spaces around it, labeled 1 through 6, and a marker. The marker starts on space 1. On each move, Mark flips a coin. If he gets tails, the marker stays where it is, and if he gets heads, he then rolls a die, with numbers 1 through 6, and moves the marker forward the number of spaces that is rolled without stopping (if the marker passes space 6, it will keep going to space 1). What is the expected numbers of moves for the marker to stop on space 6 for the first time?

This is set 4.

10. Let $P(x) = x^2 + bx + c$ be a polynomial with integer coefficients. Given that $c = 2^m$ for an integer $m < 100$, how many possible values of b are there such that $P(x)$ has integer roots?

11. A ring of six identical spheres, in which each sphere is tangent to the spheres next to it, is placed on the surface of a larger sphere so that each sphere in the ring is tangent to the larger sphere at six evenly spaced points in a circle. If the radius of the larger sphere is 5, and the circle containing the evenly spaced points has radius 3, what is the radius of each of the identical spheres?

12. Given a sequence of coin flips, such as 'HTTHTHT...', we define an *inversion* as a switch from H to T or T to H. For instance, the sequence 'HTTHT' has 3 inversions.

    Harrison has a weighted coin that lands on heads $\frac{2}{3}$ of the time and tails $\frac{1}{3}$ of the time. If Harrison flips the coin 10 times, what is the expected number of inversions in the sequence of flips?

This is set 5.

13. What is the largest prime factor of $33^4 + 32^4 - 1$?

14. Compute
$$\int_e^5 \left(\left(\frac{x}{e}\right)^x + \left(\frac{e}{x}\right)^x\right) \ln x dx.$$

15. What is the maximum value of x^2y^3 if x and y are non-negative integers satisfying $x + y \leq 9$?

This is set 6.

16. Find the number of ordered triples (a, b, c) such that $a, b, c \in \{1, 2, 3, ..., 100\}$ and a, b, c form a geometric progression in that order.

17. Compute the number of $1 \leq n \leq 100$ for which $b^n \equiv a \bmod 251$ has a solution for at most half of all $1 \leq a \leq 251$.

18. Let $f(x) = x^4 + a_3x^3 + a_2x^2 + a_1x + 16$ be a polynomial with nonnegative real roots. If $(x - 2)(x - 3)f(x) = x^6 + b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + 96$, what is the minimum possible value of b_2?

This is set 7.

19. Define a *brook* as a chess piece which can move to any square which is exactly 2 positions away. Specifically, a brook at position (x, y) can move to any (x', y') with $|x' - x| + |y' - y| = 2$. What is the maximum number of brooks that can be placed on a 6×6 chessboard so that no two attack each other?

20. Determine the number of pairs (x, y) where $1 \le x, y \le 2021$ satisfying the relation
$$x^3 + 21x^2 + 484x + 6 \equiv y^2 \pmod{2022}.$$

21. Let $\triangle ABC$ be an acute triangle with orthocenter H, circumcenter O, and circumcircle Γ. Let the midpoint of minor arc BC on Γ be M. Suppose that $AHMO$ is a rhombus. If BH and MO intersect on segment AC, determine
$$\frac{[AHMO]}{[ABC]}.$$

This is set 8.

22. *Note: this round consists of a cycle, where each answer is the input into the next problem.*

    Let k be the answer to problem 24. Ariana Grande has k identical rings on a table, each of radius 1. She wishes to split these rings into necklaces, but with the added constraint that two adjacent rings on a necklace have to subtend an arc of at least 145°. How many ways are there to partition these rings into circular necklaces so that no two unlinked rings intersect or are tangent? Note that two necklaces of the same number of rings are seen as identical.

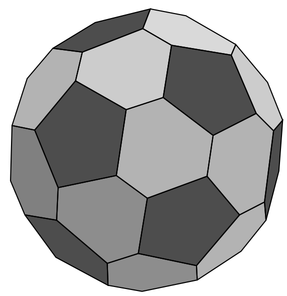
    An alternative and equivalent formulation of the problem: In how many ways can you partition a number k into unordered tuples $(k_1, k_2, \dots)$ such that $k_1 + k_2 + \cdots = k$ with each $k_i \geq 11$?

23. Let r be the answer to problem 22. Let ω_1 and ω_2 be circles of each of radius r, respectively. Suppose that their centers are also separated by distance r, and the points of intersection of ω_1, ω_2 are A and B. For each point C in space, let $f(C)$ be the the incenter of the triangle ABC. As the point C rotates around the circumference of ω_1, let S be the length of the curve that $f(C)$ traces out. If S can be written in the form $\frac{a+b\sqrt{c}}{d}\pi$ for a, b, c, d nonnegative integers with c squarefree and $\gcd(a, b, d) = 1$, then compute $a + b + c + d$.

24. Let m be the answer to problem 23. Suppose that $x_1, x_2, \dots, x_{m-1}$ are each chosen randomly and independently from the set $\{1, 2, \dots, m\}$. Then, let e_n be the expected value of $\sqrt[n]{\sum_{i=1}^{m-1} x_i^n}$. Compute $\left\lfloor \lim_{n \to \infty} e_n \right\rfloor$.

This is set 9.

25. A convex regular icosahedron has 20 faces that are all congruent equilateral triangles, and five faces meet at each vertex. A regular pentagonal pyramid is sliced off at each vertex so that the vertex lies directly above the center of the base (the diagram shows an example of an icosahedron with the pyramids sliced off, for some arbitrary size of the pyramids). The icosahedron has edge length 1. If the sliced off pyramids are identical and do not overlap, what is their largest possible total volume?



26. Consider the equation
$$\frac{a^2+ab+b^2}{ab-1}=k,$$
where $k \in \mathbb{N}$. Find the sum of all values of k, such that the equation has solutions $a, b \in \mathbb{N}, a > 1, b > 1$.

27. Compute
$$\sum_{n=0}^{1011} \frac{\binom{2022-n}{n}(-1)^n}{2021-n}.$$