1. Points A, B, C, and D lie on a circle. Let AC and BD intersect at point E inside the circle. If $[ABE] \cdot [CDE] = 36$, what is the value of $[ADE] \cdot [BCE]$? (Given a triangle $\triangle ABC$, $[ABC]$ denotes its area.)

   Answer: 36

   Solution: Let $\angle AEB = \theta$. We see that
$$[ABE] \cdot [CDE] = \frac{1}{2}\sin\theta(AE)(BE) \cdot \frac{1}{2}\sin\theta(CE)(DE).$$
   Also,
$$[ADE] \cdot [BCE] = \frac{1}{2}\sin\theta(AE)(DE) \cdot \frac{1}{2}\sin\theta(BE)(CE).$$
   Thus, $[ADE] \cdot [BCE] = [ABE] \cdot [CDE] = \boxed{36}$.

2. Let ABC be an acute, scalene triangle. Let H be the orthocenter. Let the circle going through B, H, and C intersect CA again at D. Given that $\angle ABH = 20°$, find, in degrees, $\angle BDC$.

   Answer: 110°

   Solution:

   Let E, F, G be the feet of the perpendiculars from H to lines BC, BD, AC, respectively.
   Note that E, F, G are collinear (Simpson's line), and that $BHFE, HFGD, ABEG$ are cyclic.
   Angle chasing gives $\angle BDC = \angle FHG = \angle BEF = 90 + \angle HEF = 90 + \angle ABH = 90 + 20 = 110°$.

3. $\triangle ABC$ has side lengths 13, 14, and 15. Let the feet of the altitudes from A, B, and C be D, E, and F, respectively. The circumcircle of $\triangle DEF$ intersects AD, BE, and CF at I, J, and K respectively. What is the area of $\triangle IJK$?

   Answer: 21

   Solution: First we can find that the area of $\triangle ABC$ is 84, either by noting that it can be split into 5-12-13 and 9-12-15 triangles, or using Heron's formula. Let the orthocenter of $\triangle ABC$ be H. The circumcircle of DEF is the 9-point circle of $\triangle ABC$ and thus I, J, K are the midpoints of AH, BH, CH. So, there is a homothety centered at H with factor $1/2$ that sends $\triangle ABC$ to $\triangle DEF$. Then, $[DEF] = (1/2)^2[ABC] = \boxed{21}$.

4. Let ABC be a triangle with $\angle A = \frac{135}{2}^\circ$ and $\overline{BC} = 15$. Square $WXYZ$ is drawn inside ABC such that W is on AB, X is on AC, Z is on BC, and triangle ZBW is similar to triangle ABC, but WZ is not parallel to AC. Over all possible triangles ABC, find the maximum area of $WXYZ$.

   Answer: $\frac{225\sqrt{2}}{8}$

   Solution: Let a, b, c be the lengths of sides BC, AC, and AB, respectfully, and let x be the sidelength of square $WXYZ$. Note that the given similarity condition implies that $BZ = \frac{xc}{b}$. By angle chasing, we deduce that ZXC is also similar to ABC, from which we obtain $ZC = \frac{xb\sqrt{2}}{c}$. Therefore, because $BZ + ZC = BC$, we get
$$x = \frac{a}{\frac{c}{b} + \frac{b\sqrt{2}}{c}}.$$
   Because a is fixed, x is maximized when the denominator is minimized. By AM-GM, this occurs when $\frac{c}{b} = \frac{b\sqrt{2}}{c}$ which gives a value of $2\sqrt[4]{2}$. Thus, the maximum area of the square is $x^2 = \frac{225}{4\sqrt{2}} = \boxed{\frac{225\sqrt{2}}{8}}$.

5. In quadrilateral $ABCD$, $AB = 20$, $BC = 15$, $CD = 7$, $DA = 24$, and $AC = 25$. Let the midpoint of AC be M, and let AC and BD intersect at N. Find the length of MN.

Answer: $\frac{625}{78}$

Solution: Note that $\triangle ABC$ and $\triangle ADC$ are right triangles. Since $\angle ABC + \angle ADC = 90^\circ + 90^\circ = 180^\circ$, ABCD is cyclic with circumcircle centered at M and radius $\frac{25}{2}$. Also, since $AB > BC$ and $AD > DC$, we can see that $\triangle ABD$ is acute. In $\odot M$, $\angle ABD = \angle ACD$, so $\sin\angle ABD = \frac{24}{25}$ and $\cos\angle ABD = \frac{7}{25}$. By the law of cosines, $AD^2 = AB^2 + BD^2 - 2(AB)(BD)\cos\angle ABD$ $\Rightarrow 24^2 = 20^2 + BD^2 - 2(20)(BD)\left(\frac{7}{25}\right)$. Solving the quadratic gives $BD = -\frac{44}{5}$ or 20, so we have $BD = 20$. Next, using the law of sines in $\triangle ABN$ and $\triangle ADN$ gives

$$\frac{BN}{\sin\angle BAN} = \frac{AN}{\sin\angle ABN} \Rightarrow \frac{BN}{3/5} = \frac{AN}{24/25} \Rightarrow BN = \frac{5}{8}AN$$

and

$$\frac{DN}{\sin\angle DAN} = \frac{AN}{\sin\angle ADN} \Rightarrow \frac{DN}{7/25} = \frac{AN}{4/5} \Rightarrow DN = \frac{7}{20}AN.$$

Combining this with $BN + DN = BD = 20$, we get $BN = \frac{500}{39}$ and $DN = \frac{280}{39}$. Then, $AN = \frac{8}{5}BN = \frac{800}{39}$. Finally, the $MN = AN - AM = \frac{800}{39} - \frac{25}{2} = \boxed{\frac{625}{78}}$.

6. Let the incircle of $\triangle ABC$ be tangent to AB, BC, AC at points M, N, P, respectively. If $\measuredangle BAC = 30^\circ$, find $\frac{[BPC]}{[ABC]} \cdot \frac{[BMC]}{[ABC]}$, where $[ABC]$ denotes the area of $\triangle ABC$.

Answer: $\frac{1}{2} - \frac{\sqrt{3}}{4}$

Solution: If u, w denote the distance between P and M to BC respectively, we need to compute $\frac{uw}{h_a^2}$. By Thales' theorem, we have that $\frac{u}{h_a} = \frac{CP}{CA} = \frac{p-c}{b}$ and $\frac{w}{h_a} = \frac{BM}{BA} = \frac{p-b}{c}$, where p is the semiperimeter of $\triangle ABC$. Let I be the incenter of ABC, and assume standard notation for sides and angles. Then, from the law of sines for BMI, we have that $p - b = BI\cos\frac{\beta}{2}$. From ABI, $BI = \frac{c}{\cos\frac{\gamma}{2}}\sin\frac{\alpha}{2}$, and so we get $\frac{p-b}{c} = \frac{\cos\frac{\beta}{2}}{\cos\frac{\gamma}{2}}\sin\frac{\alpha}{2}$. Analogously, $\frac{p-c}{b} = \frac{\cos\frac{\gamma}{2}}{\cos\frac{\beta}{2}}\sin\frac{\alpha}{2}$, and hence, $\frac{uw}{h_a^2} = \sin\frac{\alpha}{2}^2$. Plugging in $\alpha = 30$, we get $\frac{1}{2} - \frac{\sqrt{3}}{4}$.

7. $\triangle ABC$ has side lengths $AB = 20$, $BC = 15$, and $CA = 7$. Let the altitudes of $\triangle ABC$ be AD, BE, and CF. What is the distance between the orthocenter (intersection of the altitudes) of $\triangle ABC$ and the incenter of $\triangle DEF$?

Answer: 15

Solution: Note that $7^2 + 15^2 = 274 < 400 = 20^2$, so $\triangle ABC$ is obtuse, which means the orthocenter, which we will denote H, lies outside $\triangle ABC$. We have $\angle ADB = \angle BEA = 90^\circ$, so quadrilateral $ADEB$ is cyclic. In $(ADEB)$, we can see that $\angle AED = ABD$. Also, since $\angle AFH = \angle AEH = 90^\circ$, quadrilateral $AFEH$ is cyclic. In $(AFEH)$, we can see that $\angle AEF = \angle AHF = 90^\circ - \angle HAF = 90^\circ - (90^\circ - ABD) = \angle ABD$. So, $AED = AEF$, which means AE bisects $\angle DEF$. Similarly, we can show that BD bisects $\angle EDF$. Therefore, the incenter of $\triangle DEF$ is the intersection of AE and BD, which is C.
We see that $CF = AC\sin\angle BAC$. Also,

$$HF = AF\tan\angle HAF = (AC\cos\angle BAC)\tan(90^\circ - \angle ABC) = AC\cos\angle BAC\cot\angle ABC.$$

Now, we want to calculate

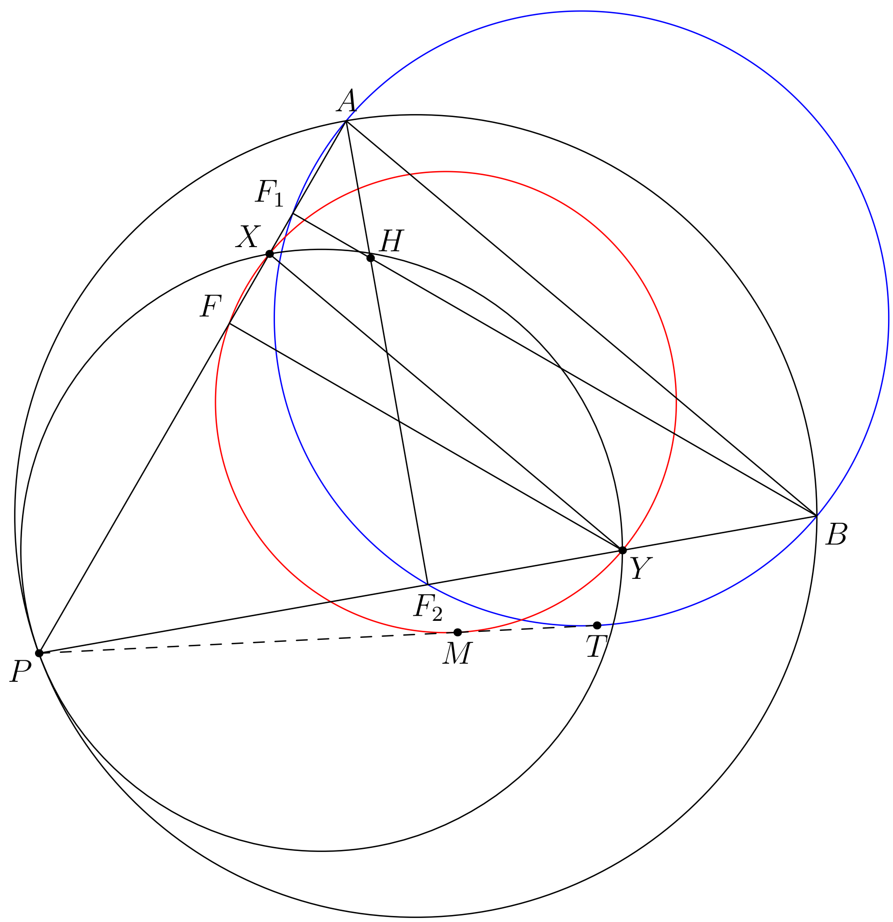
$$HC = HF - CF = AC\cos\angle BAC\cot\angle ABC - AC\sin\angle BAC.$$

Using the law of cosines, we have $\cos\angle BAC = \frac{7^2+20^2-15^2}{2\cdot 7\cdot 20} = \frac{4}{5}$, so $\sin\angle BAC = \frac{3}{5}$. Also, $\cos\angle ABC = \frac{15^2+20^2-7^2}{2\cdot 15\cdot 20} = \frac{24}{25}$, so $\sin\angle ABC = \frac{7}{25}$ and $\cot\angle ABC = \frac{24}{7}$. Finally, we have $HC = AC(\cos\angle BAC \cot\angle ABC - \sin\angle BAC) = 7\left(\frac{4}{5}\cdot\frac{24}{7} - \frac{3}{5}\right) = \boxed{15}$.

8. Let Γ and Ω be circles that are internally tangent at a point P such that Γ is contained entirely in Ω. Let A, B be points on Ω such that the lines PB and PA intersect the circle Γ at Y and X respectively, where $X, Y \neq P$. Let O_1 be the circle with diameter AB and O_2 be the circle with diameter XY. Let F be the foot of Y on XP. Let T and M be points on O_1 and O_2 respectively such that TM is a common tangent to O_1 and O_2. Let H be the orthocenter of $\triangle ABP$. Given that $PF = 12, FX = 15, TM = 18, PB = 50$, find the length of AH.

Answer: $\frac{\mathbf{750}}{\sqrt{\mathbf{481}}}$

Solution:



Since Γ and Ω are tangent at P, there exists a homothety centered at P which maps Γ to Ω. Denote this homothety by h. Let k be its common ratio. We can see that A, B must be the image of the points X, Y under h respectively. Thus, $h(O_2) = O_1$. Therefore, the common tangents to O_1 and O_2 intersect at P. Hence, P, M, T are collinear, since $h(M) = T$.

Observe that the power of the point P with respect to O_2 is given by $PF \cdot PX = 324$. However, PM is tangent to O_1, and thus the power of P with respect to O_1 is $PM^2 = PF \cdot PX = 324$.

This gives us that $PM = \sqrt{324} = 18$ and $PT = 18+18 = 36$. Thus, the common ratio of the homothety is $k = \dfrac{PT}{PM} = 2$. Let F_1 be the foot of B on AP. Then, we have that $PF_1 = 2 \cdot PF = 24$. Additionally, we can see that $PA = 2 \cdot PX = 54$. Therefore, $AF_1 = 30$.

Similarly, we can compute PY since $PY = \frac{1}{2} \cdot PB = 25$. Therefore, by the Pythagorean theorem, we obtain

$$FY = \sqrt{PY^2 - PF^2} = \sqrt{25^2 - 12^2} = \sqrt{481}.$$

Let F_2 be the foot of A onto PB. Then, H is the intersection of AF_2 and BF_1. Now observe that $\angle F_1AH = \angle PAF_1 = 90^\circ - \angle APF_2 = 90^\circ - \angle FPY = \angle FYP$. Thus, by AA, we have $\triangle AF_1H \sim \triangle YFP$. Thus,

$$\frac{AF_1}{AH} = \frac{FY}{PY} \implies AH = \frac{AF_1 \cdot PY}{FY} = \frac{30 \cdot 25}{\sqrt{481}}.$$

Thus, $AH = \frac{750}{\sqrt{481}}$.

9. The bisector of $\angle BAC$ in $\triangle ABC$ intersects BC in point L. The external bisector of $\angle ACB$ intersects $\overrightarrow{BA}$ in point K. If the length of AK is equal to the perimeter of $\triangle ACL$, $LB = 1$, and $\angle ABC = 36^\circ$, find the length of AC.

   Answer: 1

   Solution: Let T be a point on $\overrightarrow{AC}$ such that $AT = AK$. Then, $\measuredangle ATK = \measuredangle AKT = \frac{\alpha}{2}$. Now let $B' \in \overrightarrow{LB}$ such that $LB' = AL$. We then have $CB' = CT$ and since $\measuredangle B'CK = \measuredangle TCK = 90 + \frac{\gamma}{2}$, we attain $\triangle KCB' \cong KCT$. Therefore, $\measuredangle CB'K = \frac{\alpha}{2}$. If B' is between L and B, then $\measuredangle CB'K < \measuredangle CB'A = \measuredangle LAB' < \frac{\alpha}{2}$ which is a contradiction. Similarly, if B is between L and B', we get that $\measuredangle CB'K > \measuredangle CB'A = \measuredangle LAB' > \frac{\alpha}{2}$, which is also a contradiction. Therefore, $B' \equiv B$ and $\measuredangle CBA = \frac{\alpha}{2} = 36^\circ$. We now get $\alpha = 72^\circ$ and so, $LB = AL = AC = 1$, as desired.

10. Let $ABCDEFGH$ be a regular octagon with side length $\sqrt{60}$. Let $\mathcal{K}$ denote the locus of all points K such that the circumcircles (possibly degenerate) of triangles HAK and DCK are tangent. Find the area of the region that $\mathcal{K}$ encloses.

    Answer: $(240 + 180\sqrt{2})\pi$

    Solution: Let the side length of our octagon be s. We will plug in $\sqrt{60}$ later. Consider the radical center of the circles $(ABCDEFGH), (HAK)$, and (DCK). Note that it is the intersection of lines DC and HA. Let this intersection point be I. Then it becomes clear that $\mathcal{K}$ is a circle centered at I, since we have that

$$KI^2 = DI \cdot CI \implies KI \text{ is fixed,}$$

by Power of a Point. It is also not hard to see that any point K on this circle will work. Now we need only compute $DI \cdot CI$. Note that from similar triangles HDI and ACI we have

$$\frac{HD}{AC} = \frac{DI}{CI} = \frac{s + CI}{CI} \implies CI = \frac{AC}{HD - AC}s$$

Then from the property that ACE is an isosceles right triangle and that $AE = HD$ we have that $HD = \sqrt{2}AC$, and so

$$CI = \frac{s}{\sqrt{2} - 1} = s(1 + \sqrt{2})$$

and then because $DI = CI + s$ we have that

$$DI \cdot CI = s^2(1+\sqrt{2})(2+\sqrt{2}) = s^2(4+3\sqrt{2})$$

hence the area of $\mathcal{K}$ is $s^2(4+3\sqrt{2})\pi$. Substituting $s^2 = 60$ we get that the area of $\mathcal{K}$ is $\boxed{(240+180\sqrt{2})\pi}$.