1. There are natural numbers a and b, where b is square-free, for which we can write

$$20+\cfrac{1}{20+\cfrac{1}{20+\cfrac{1}{20+\cdots}}}=a+\sqrt{b}.$$

What is $a+\sqrt{b}$?

Answer: $\mathbf{10+\sqrt{101}}$

Solution: Let $u=20+\frac{1}{20+\frac{1}{20+\frac{1}{20+\cdots}}}$. Then $u=20+1/u$. For non-zero u, we arrive at the equation $u^2=20u+1$. We solve this to find that $10+\sqrt{101}$ is the solution.

2. Suppose there are five cars and three roads ahead. Each car selects a road to drive on uniformly at random. Every car adds a one minute delay to the car behind them. What is the expected delay of a car selected uniformly at random from the five cars? (For example, if cars $1,2,3$ go on road A in that order and cars $4,5$ go on road B, then the delays for the cars are $0,1,2,0,1$ respectively.)

Answer: $\mathbf{\frac{2}{3}}$

Solution: Smart sol: For any car, there is a $1/3$ chance for a car in front of it to be in the same lane. There are a total of $0+1+2+3+4=10$ cars "in front" of other cars, so the total expected delay is $10/3$. Finally, there are 5 cars total so the expected delay of a particular car should be $(10/3)/5=2/3$.

Bashy sol: We will work through different cases of what the car arrangements can be. Without loss of generality, we list lanes in the order of most cars to least cars, and discuss the actual number of possibilities while doing each case. Note that the total delay is independent of which exact car is where, as long as the overall distribution of cars is the same.

Also, to simplify calculations, we can use linearity of expectation to calculate the total delay in each case and put them together in a weighted average at the very end.

Case 1: 5/0/0
In this case, there are 3 possible arrangements, as there are 3 ways to choose the lane with 5 cars. Here, the total delay is $0+1+2+3+4=10$

Case 2: 4/1/0
Here, there are 3 ways to choose the road with 4 cars, 2 ways to choose the road with 1 car, and $5C1=5$ ways to choose the car that is all sad and alone. Thus, this there are 30 different ways for this arrangement to happen. Here, the total delay is $0+1+2+3+0=6$.

Case 3: 3/2/0
As in case 2, there are $3\cdot 2$ ways to choose our two roads; then there are $5C2=5C3=10$ ways to choose the two or three cars to form a lane. Hence, there are a total of 60 different arrangements of this case. Here, the total delay is $0+1+2+0+1=4$.

Case 4: 3/1/1
We only need to select the lane with 3 cars, so lane selection contributes a factor of 3. Then, there are $5C2=10$ ways to select the two single cars, and finally 2 ways to place them into the two lanes. Thus, there are 60 arrangements in this case. Here, the total delay is $0+1+2+0+0=3$.

Case 5: 2/2/1
There are $5C2=10$ ways to choose the two cars in the first two car lane and $3C2=3$ ways o choose the two cars for the second two car lane. Finally, there are 3 ways to choose the

lane that has a single car for a total of $3 \cdot 3 \cdot 10 = 90$ arrangements. Here, the total delay is $0 + 1 + 0 + 1 + 0 = 2$.

For a quick sanity check, note that $3+30+60+60+90 = 243 = 3^5$ which is equal to the number of arrangements we get if we let each car pick between the three roads one at a time. Thus, as there are 5 cars, and the chance to be each car is $1/5$, the overall expected delay is

$$\frac{1}{5}\frac{3 \cdot 10 + 30 \cdot 6 + 60 \cdot 4 + 60 \cdot 3 + 90 \cdot 2}{243} = \frac{(5 \cdot 3)(2 + 12 + 20 + 12 + 12)}{5 \cdot 3^5} = \frac{54}{3^4} = \frac{2}{3}.$$

3. Compute
$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \cdots + \frac{1}{17 \cdot 19}.$$

Answer: $\mathbf{\frac{9}{19}}$

Solution: Since $\frac{1}{n*(n+2)} = (\frac{1}{n} - \frac{1}{n+2}) * \frac{1}{2}$, we can use telescoping series to determine the sum to be $(1 - \frac{1}{3}) * \frac{1}{2} + (\frac{1}{3} - \frac{1}{5}) * \frac{1}{2} + \cdots + (\frac{1}{17} - \frac{1}{19}) * \frac{1}{2} = (1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \cdots + \frac{1}{17} - \frac{1}{19}) * \frac{1}{2} = (1 - \frac{1}{19}) * \frac{1}{2} = \frac{9}{19}$

4. Consider an acute angled triangle $\triangle ABC$ with sides of length a, b, c. Let D, E, F be points (distinct from A, B, C) on the circumcircle of $\triangle ABC$ such that: $AD \perp BC$, $BE \perp AC$, $CF \perp AB$. What is the ratio of the area of the hexagon $AECDBF$ to the area of the triangle $\triangle ABC$?

Answer: 2

Solution:

(a) We first show that triangles $\triangle HBC \cong \triangle DBC$. This follows from the following properties:

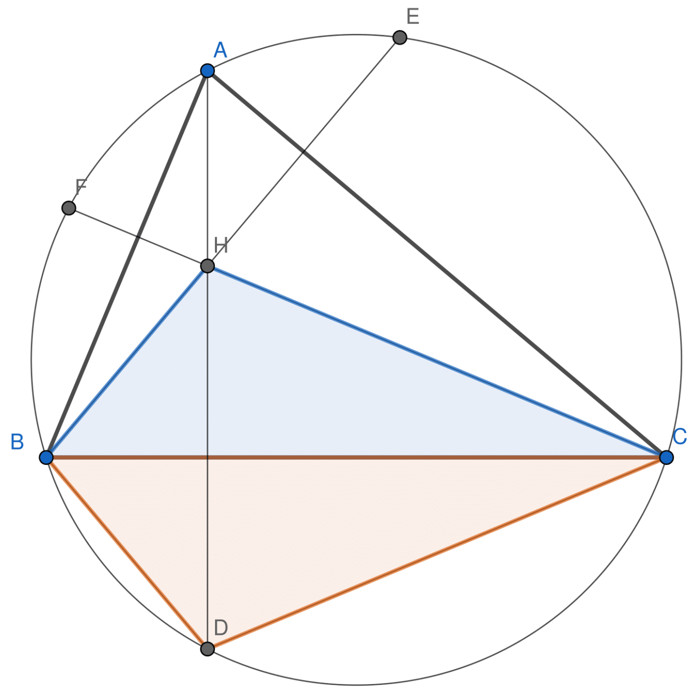
$$\begin{aligned}\angle HBC &= 90 - \angle ACB \\ &= \angle DAC \\ &= \angle DBC\end{aligned}$$

Similarly we can show that $\angle HCB = \angle DCB$. This completes the congruence proof.

(b) We can similarly show that: $\triangle HAB \cong \triangle FAB$ and $\triangle HCA \cong \triangle ECA$.

(c) It is easy to then obtain the area of the hexagon $AECDBF$ to be:

$$2A(\triangle ABC) = 2\sqrt{s(s-a)(s-b)(s-c)}$$



5. Let $ABCD$ be a trapezoid with AB parallel to CD and $AB = BC = AD = 8$. Side CD is extended past D to a point E so that $DE = 8$ and $CD = AE$. What is the length of CD?

   Answer: $\mathbf{4 + 4\sqrt{5}}$

   Solution: Note that $AB = DE = 8$, $\angle BAD = \angle ADE$ (since $AB \parallel CD$), and $AD = AD$, so $\triangle ABD \cong \triangle DAE$, which gives $BD = AE = CD$. Let $\angle BDA = a$. Then, $\angle BAD = 180^\circ - 2a$ and $\angle CBD = \angle CBA - \angle DBA = \angle BAD - \angle BDA = 180^\circ - 2a - a = 180^\circ - 3a$. Also, $\angle BDC = \angle DBA = a$, so $\angle ADC = \angle BDA + \angle CDB = a + a = 2a$.
   We have $\angle BCD = \angle ADC = 2a$. Since $\triangle BCD$ is isosceles, we have $\angle BCD = \angle CBD \Rightarrow 180^\circ - 3a = 2a \Rightarrow a = 36^\circ$. We recognize that $\triangle BCD$ has angles 36°, 36°, and 72°, so its side lengths are in the ratio $\frac{1+\sqrt{5}}{2}$. We have $CD = 8\left(\frac{1+\sqrt{5}}{2}\right) = \boxed{4 + 4\sqrt{5}}$.

6. What are the last two digits of $2022^{2022^{2022}}$?

   Answer: **56**

   Solution: We wish to compute $2022^{2022^{2022}} \bmod 100$, or in other words, we wish to compute the exponent modulo $\varphi(100) = 40$. We can go one step further: in particular,

$$2022^{2022^{2022}} \bmod 100 = 2022^{2022^{2022} \bmod 40} \bmod 100 = 2022^{2022^{2022 \bmod 16} \bmod 40} \bmod 100$$

   Since $2016 \equiv 0 \bmod 16$, we have to compute $2022^6 \bmod 40$. Notice that $2022^6 \equiv 0 \bmod 8$, and $2022^6 \equiv 4 \bmod 5$. So, $2022^6 \equiv 24 \bmod 40$. Therefore, it suffices to compute $2022^{24} \equiv 22^{24} \bmod 100$. To this end, again note that $22^{24} \equiv 0 \bmod 4$ and $22^{24} \equiv 22^4 \equiv (-3)^4 \equiv 6 \bmod 25$. So, $22^{24} \equiv \boxed{56} \bmod 100$.

7. On December 9, 2004, Tracy McGrady scored 13 points in 33 seconds to beat the San Antonio Spurs. Given that McGrady never misses and that each shot made counts for 2, 3, or 4 points, how many shot sequences could McGrady have taken to achieve such a feat assuming that order matters?

Answer: 52

Solution: We first narrow down our search by noticing that since 13 is an odd number, each sequence must have an odd number of 3-pointers to reach 13. So, we must have 1 or 3 3-pointers.

If there is 1 3-pointer, then the number of 2-pointers, 3-pointers, and 4-pointers written as ordered triples can be (5, 1, 0), (3, 1, 1), or (1, 1, 2). This gives us $\frac{6!}{5!1!0!}+\frac{5!}{3!1!1!}+\frac{4!}{1!1!2!}=6+20+12=38$ possibilities.

If there are 3 3-pointers, then the number of 2-pointers, 3-pointers, and 4-pointers written as ordered triples can be (2, 3, 0) or (0, 3, 1). This gives us $\frac{5!}{2!3!0!}+\frac{4!}{0!3!1!}=10+4=14$ possibilities.

In total, we have $38+14=\boxed{52}$ possibilities.

8. Stanford has a new admissions process that it would like to test out on the Stanford Class of 2027. An admissions officer starts by ordering applicants $1, 2, \ldots,$ and 2022 in a circle with applicant 1 being after applicant 2022. Then, starting with applicant 1, the admissions officer removes every 2023rd applicant. What is the number of the applicant removed in the 49th iteration?

Answer: 1225

Solution: First, we notice that for the first 63 applicants, the applicant to be removed is a triangular number. Then, the 49th triangular number is 1225.

9. Over all pairs of real numbers (x, y) with $x^2+y^2=1$, let m be the maximum value of $4xy-8xy^3$. At what values of x is m attained? (List your answers separated by commas, in any order.)

Answer: $\frac{\sqrt{2\pm\sqrt{2}}}{2}, -\frac{\sqrt{2\pm\sqrt{2}}}{2}$

Solution: Let $x=\cos\theta$ and $y=\sin\theta$ for some angle θ. Then, note that

$$\begin{aligned}
4xy-8xy^3 &= 4\cos\theta\sin\theta-8\cos\theta\sin^3\theta \\
&= 4\cos\theta\sin\theta(1-2\sin^2\theta) \\
&= 4\cos\theta\sin\theta(\cos^2\theta-\sin^2\theta) \\
&= 2\sin(2\theta)\cos(2\theta) \\
&= \sin(4\theta).
\end{aligned}$$

$\sin(4\theta)$ is maximized when $4\theta=\pi/2$, so we have $x=\cos(\pi/8)=\sqrt{\frac{1-\cos(\pi/4)}{2}}=\frac{\sqrt{2-\sqrt{2}}}{2}$. The maximum is also achieved when $4\theta=5\pi/2, 9\pi/2,$ and $13\pi/2$, which yields 3 other solutions (increasing θ further repeats solutions that have already been found). So, there are 4 solutions in total and our answer is $\boxed{\frac{\sqrt{2\pm\sqrt{2}}}{2}, -\frac{\sqrt{2\pm\sqrt{2}}}{2}}$. (Note: During the tournament [not the international run], any nonempty subset of the solutions was accepted.)

10. You need to bike to class but don't know where you parked your bike. There are two bike racks, A and B. There is a $1/5$ chance for your bike to be at A; it takes one minute to walk to A and four minutes to bike from A to class. Then, there is a $4/5$ chance for your bike to be at B; it takes three minutes to walk to B and five minutes to bike from B to class. However, if your choice is wrong, you need to walk from your original choice A or B to the other, which takes four minutes, before departing to class from there.

    Suppose you only care about getting to class on time. For a some interval of minutes before class, going to bike rack B first gives a strictly higher chance of making it to class on time. How many minutes long is that interval (i.e. an interval of 15 minutes before class to 21 minutes before class has length 6)?

    Answer: 2

    Solution: If you go to bike rack A first, then there is a $1/5$ chance that your bike is there and you get to class in $1 + 4 = 5$ minutes. Otherwise, there is a $4/5$ chance that your bike is not there, and you get to class in $1 + 4 + 5 = 10$ minutes.

    If you go to bike rack B first, then there is a $4/5$ chance that your bike is there and you get to class in $3+5 = 8$ minutes. Otherwise, there is a $1/5$ chance that your bike is not there, and you get to class in $3 + 4 + 4 = 11$ minutes.

    If there are less than 5 minutes before class, then both options give you no chance of getting to class on time. If there are between 5 and 8 minutes before class, then going to bike rack A gives you a $1/5$ chance of making it to class on time, while going to bike rack B gives you no chance of making it to class on time. If there are between 8 and 10 minutes before class, then going to bike rack A still gives you a $1/5$ chance of making it to class on time, while going to bike rack B now gives you a $4/5$ chance of making it to class on time. If there are more than 10 minutes before class starts, then going to bike rack A will always lead to you making it to class on time, which means going to bike rack B will never give you a strictly greater chance of making it to class on time.

    Thus, going to bike rack B is better only if there are between 8 and 10 minutes before class, which is an interval of 2 minutes.

11. Suppose that x and y are complex numbers satisfying the relations

$$\begin{aligned} x^2 + y^2 &= 11 \\ x^3 + y^3 &= 20 \\ x^4 + y^4 &= 23 \\ x^5 + y^5 &= -25. \end{aligned}$$

    Compute $x^6 + y^6$.

    Answer: $-$286

    Solution: Note that $(x^n + y^n)(x + y) = x^{n+1} + y^{n+1} + xy(x^{n-1} + y^{n-1})$. If we make the substitutions $u = x + y$ and $v = xy$, we can plug in $n = 3, 4$ to get the system of equations

$$\begin{aligned} 20u &= 23 + 11v \\ 23u &= -25 + 20v \end{aligned}$$

    Solving gives us $u = 5$ and $v = 7$. Thus, we have $x^6 + y^6 = -25 \cdot 5 - 23 \cdot 7 = \boxed{-286}$.

12. Let $S_n = \sum_{j=1}^{n} j^3$. Find the smallest positive integer n such that the last three digits of S_n are all zero.

Answer: 24

Solution: We know that $S_n = \sum_{j=1}^{n} j^3 = (\frac{n(n+1)}{2})^2$ and we want the smallest positive integer n such that $1000 \mid S_n$. This means $4000 \mid n^2(n+1)^2$. Note that $4000 = 2^5 \cdot 5^3$. Since only one of n and $(n+1)$ is even, either $2^5 \mid n^2$ or $2^5 \mid (n+1)^2$. So, either $8 \mid n$ or $8 \mid n+1$. Similarly, only one of n and $n+1$ is divisible by 5 and so, either $5^3 \mid n^2$ or $5^3 \mid (n+1)^2$. This occurs when $25 \mid n$ or $25 \mid n+1$. A quick observation suggests that $n = \boxed{24}$ is the smallest positive integer satisfying this requirement.

13. Consider right triangles with two legs on the x and y axes with hypotenuse tangent to the line $y = 2022/x$ (the slope of this line at $x = a$ is $-2022/a^2$). If two tangent points are chosen uniformly at random on the curve $y = 2022/x$ restricted to $1/2022 \leq x \leq 2022$, what is the expected ratio of the area of the triangle with larger y-intercept to the area of the triangle with lesser y-intercept?

Answer: 1

Solution: Suppose the point $(a, 2022/a)$ is chosen. Then, the slope of the line is $-2022/a^2$ so the equation of this line in point-slope form is

$$y - \frac{2022}{a} = \frac{-2022}{a^2}(x - a).$$

Solving for intercepts by plugging in x or y equals to 0, the x-intercept is $(2a, 0)$ while the y-intercept is $(0, 4044/a)$. Thus, the area of this triangle is

$$\frac{1}{2} \cdot 2a \cdot \frac{4044}{a} = 4044$$

so area is independent of the point chosen. Thus, the expected ratio in area is just one.

14. A tiebreaker in tennis is played until a player has seven or more points and is winning by at least two points. Players A and B take turns serving 2 points each in the order ABBAABBAA.... Whoever serves a point has a 70% chance of winning that point. Given that A served first and that the score is currently 4-4, what is the probability that A wins the tiebreaker?

Answer: $\mathbf{\frac{1}{2}}$

Solution: Note that the if a player were to win after an odd number of points has been played, that player would still win after an additional point has been played. Thus we can wait to evaluate the winner of the tiebreak after an even number of points has been played. Note that in any increment of 2 points, both A and B serve once, so it is symmetric. Thus the probability A wins is $\boxed{\frac{1}{2}}$

15. What is the unique perfect cube c of the form $c = k^3 + k^2 + 11k + 1$ for some strictly positive integer k?

Answer: 125

Solution: Clearly, $c = (k + n)^3$ for some n. Note that n cannot be too large; for $n = 2$ we already have that $(k + 2)^3$ is strictly greater than $k^3 + k^2 + 11k + 1$. Setting $c = (k + 1)^3$ we need to solve

$$(k + 1)^3 = k^3 + 3k^2 + 3k + 1 = k^3 + k^2 + 11k + 1 \implies 2k^2 = 8k \implies k = 4$$

so substituting this back into $k^3 + k^2 + 11k + 1$ gives the unique square to be 125.

16. Consider the parabola $y = x^2$. Let a circle centered at $(0, a)$ be tangent to the parabola at (b, c) such that $a+c = (2b-1)(2b+1)$. If $a > 0$, find the area of the finite region between the parabola and the circle.

Answer: $\boldsymbol{\frac{3\sqrt{3}}{4} - \frac{\pi}{3}}$

Solution: If $b = 0$, then $c = 0$ and $a = -1$, which isn't allowed, so $b \neq 0$. WLOG $b > 0$. Denote $(0, a)$ by C and (b, c) by P. Then the slope of CP is $\frac{c-a}{b}$, and the tangent at P has slope $2b$. If the circle is tangent to the parabola at P, then its radius CP must be perpendicular to the tangent of the parabola, hence $\frac{c-a}{b} = -\frac{1}{2b}$, i.e. $c = b^2 = a - \frac{1}{2}$. We are given $4b^2 - 1 = a + c = 2b^2 + \frac{1}{2}$, so $b = \frac{\sqrt{3}}{2}$, which implies $c = \frac{3}{4}, a = \frac{5}{4}$.

Let Q be $(b, 0)$ and let O be $(0, 0)$. The desired region is symmetric about the y-axis, so consider the right portion. That area can be calculated as the area of trapezoid $OCPQ$ minus the area under the parabola from $x = 0$ to b and the sector of the circle created by sweeping the initially downward radius up to the radius CP.

The area of the trapezoid is $\frac{1}{2}(a + c)b = \frac{(4b^2-1)b}{2}$. The area under the parabola is $\frac{b^3}{3}$. Let the radius of the circle be r. We see from Pythagoras that $r^2 = b^2 + (a - c)^2 = b^2 + \frac{1}{4}$. The angle of the relevant sector is $\tan^{-1}(\frac{b}{c-a}) = \tan^{-1}(2b) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$. The answer is then

$$2\left(\frac{(4b^2-1)b}{2} - \frac{b^3}{3} - \frac{\pi/3}{2\pi}\pi(b^2 + \frac{1}{4})\right) = \boxed{\frac{3\sqrt{3}}{4} - \frac{\pi}{3}}.$$

17. Three cities X, Y and Z lie on a plane with coordinates (0, 0), (200, 0) and (0, 300) respectively. Town X has 100 residents, town Y has 200, and town Z has 300. A train station is to be built at coordinates (x, y), where x and y are both integers, such that the overall distance traveled by all the residents is minimized. What is (x, y)?

Answer: $\boldsymbol{(0, 300)}$

Solution: We claim that the station should be at Z. Suppose that it's at some location W different from Z. The change in distance traveled

$$\begin{aligned}
&= 300ZW + 200YW + 100XW - 200YZ - 100XZ \\
&= 100(ZW + WX - ZX) + 200(ZW + WY - ZY) \\
&> 0
\end{aligned}$$

by triangle inequality. Thus for the distance to be minimized, we want $x = 0, y = 300$.

18. What is the cardinality of the largest subset of $\{1, 2, ..., 2022\}$ such that no integer in the subset is twice another?

Answer: **1348**

Solution 1: Suppose that we were actually solving the problem for 2047 as the upper bound instead of 2022. For each odd number $m \leq 2047$, consider the chain $[m, 2m, 4m, \ldots, 2^k m]$ such that $2^k m \leq 2047$ and $2^{k+1}m > 2047$. Now, the problem reduces to not having adjacent elements in a chain (chains do not interact). Notice that if a chain has length k, then at most $\lceil \frac{k}{2} \rceil$ elements can appear in our final subset.

Now we sum over chains. In particular, we split chains based on between which powers of 2 they begin. So, for example, all chains that start in $[1024, 2047]$ will have 1 element in our final set, and similarly for $[512, 1023]$.

So, our answer would be

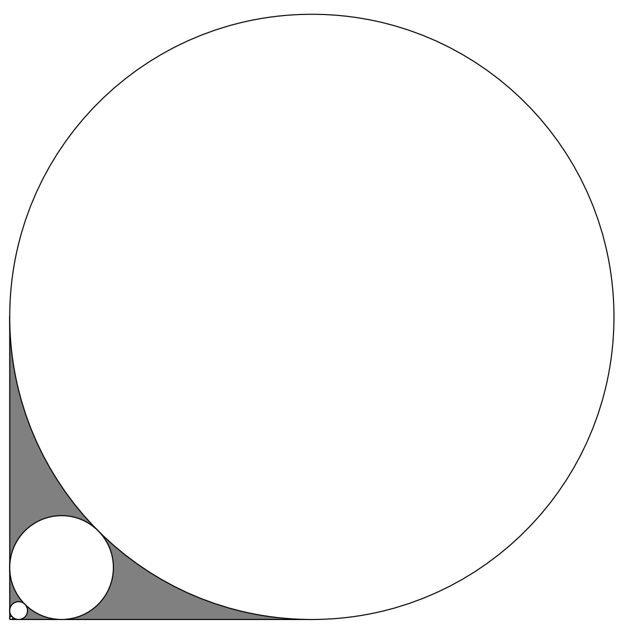
$$(2^9 + 2^8) \cdot 1 + (2^7 + 2^6) \cdot 2 + \ldots + 6 = 1365.$$

The last 6 is for the chain beginning at $m = 1$.

Now notice that this has overcount. Specifically, it overcounts (by one) any odd length chain ending in $[2023, 2047]$ (even length chains can always skip this interval). There are 13 odd numbers in this interval. There are also 3 numbers of the form $4m$, and 1 number of the form $16m$. This yields a final answer of $1365 - 13 - 3 - 1 = \boxed{1348}$.

Solution 2: An equivalent way of counting the size of the largest possible subset, after reasoning through the logic of solution 1, is to add up the number of odd numbers m from 1 to 2022, then numbers of the form $4m$, then numbers of the form 4^2m, and so on. This gives us
$2022/2 = 1011 \quad (m)$,
$\lfloor 2022/4 \rfloor = 505 \to \lceil 505/2 \rceil = 253 \quad (4m)$,
$\lfloor 2022/16 \rfloor = 126 \to \lceil 126/2 \rceil = 63 \quad (4^2m)$,
$\lfloor 2022/64 \rfloor = 31 \to \lceil 31/2 \rceil = 16 \quad (4^3m)$,
$\lfloor 2022/256 \rfloor = 7 \to \lceil 7/2 \rceil = 4 \quad (4^4m)$,
$\lfloor 2022/1024 \rfloor = 1 \to \lceil 1/2 \rceil = 1 \quad (4^5m)$.
In total we have $1011 + 253 + 63 + 16 + 4 + 1 = \boxed{1348}$.

19. Let C_1 be the circle of radius 1 centered at $(1, 1)$ on the xy plane. Define C_n to be the circle tangent to C_{n-1}, $x = 0$, and $y = 0$. What is the area of the shaded region?



Answer: $\mathbf{\frac{8+2\pi-3\pi\sqrt{2}}{8}}$

Solution: Let r_n be the radius of C_n. We can see that $r_{n-1}(\sqrt{2}-1) = r_n(\sqrt{2}+1)$. So, we have a geometric progression of

$$r_n = \left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)^{n-1} = (\sqrt{2}-1)^{2n-2}.$$

Then the area of circles

$$\sum_{n=2}^{\infty} C_n = \sum_{n=2}^{\infty} \pi\left(\sqrt{2}-1\right)^{4n-4} = \pi\frac{(\sqrt{2}-1)^4}{1-(\sqrt{2}-1)^4} = \pi\left(\frac{3\sqrt{2}}{8}-\frac{1}{2}\right).$$

Thus, we have that the shaded region has area:

$$\frac{1}{2}-\left(\frac{\pi}{4}-\frac{1}{2}\right)-\pi\left(\frac{3\sqrt{2}}{8}-\frac{1}{2}\right) = \boxed{\frac{8+2\pi-3\pi\sqrt{2}}{8}}$$

20. For each positive integer n, define $f(n)$ to be the number of positive integers m such that $\gcd(m,n)^2 = \operatorname{lcm}(m,n)$. Compute the smallest n such that $f(n) > 10$.

Answer: 44100

Solution: Note that for any $n = p_1^{e_1}\dots p_k^{e_k}$, $f(n) = 2^s$ where s is the number of e_i that are even. Therefore, for $f(n) > 10$, s must be at least 4, which means n is minimized when it equals $2^2\cdot 3^2\cdot 5^2\cdot 7^2 = 210^2 = 44100$.

21. Let $\triangle ABC$ be a triangle with $\overline{AB} = 20$ and $\overline{AC} = 22$ and circumcircle ω. Let H be the orthocenter of the triangle, and let $\overline{AA'}$ be a diameter of ω. Suppose that $\overline{CA'}$ intersects $\overline{AH}$ at T. If $BHCT$ is cyclic then the sum of all possible lengths of $\overline{BC}$ can be expressed in the form $a\sqrt{b}$ where a and b are integers and b is square-free. What is $a\sqrt{b}$?

Answer: $\mathbf{22\sqrt{2}}$

Solution: Since AA' is a diameter, we have $\angle TCA = \angle A'CA = 90^\circ$. Note that $\angle CAT = 90^\circ - \angle ACB = \angle HBC = \angle HTC = \angle ATC$, so we actually have that $\triangle TCA$ is an isosceles right triangle with internal angle bisector CB. Hence $\angle BCA = 45^\circ$. Let the length of BC be x. Now from the Law of Cosines we have

$$20^2 = x^2 + 22^2 - 44x\cos 45^\circ \implies x^2 - 22\sqrt{2}x + 84 = 0$$

and since the discriminant is positive, the sum of the possible lengths is $22\sqrt{2}$ and our answer is $22\sqrt{2}$.

22. Let $ABCA_1B_1C_1$ be a right regular triangular prism with triangular faces $\triangle ABC$ and $\triangle A_1B_1C_1$ and edges $\overline{AA_1}, \overline{BB_1}, \overline{CC_1}$. A sphere is tangent to sides $\overline{AB}, \overline{BC}, \overline{AC}$ at points M, N, P and to the plane that the triangle $\triangle A_1B_1C_1$ is in at point Q. Let $\measuredangle MPQ = 45^\circ$ and the distance between lines $\overline{MP}$ and $\overline{NQ}$ be equal to 1. Find the the side length of the base of the prism.

Answer: 4

Solution: Denote the incircle of ABC with point I. It is trivial to show that I, O (center of the sphere) and Q are collinear and form a line perpendicular to the bases. Further, by the theorem of the three perpendiculars, we have that $NQ \perp BC$.

Denote the side length of the base with a. We have that $MP = a/2$. Therefore, from isosceles $\triangle MPQ$ with angle at the base $45°$, $MQ = PQ = \frac{a\sqrt{2}}{4}$. Hence, $NQ = \frac{a\sqrt{2}}{4}$ as well.

Let K denote the midpoint of MP, and let KT be perpendicular to NQ, $K \in NQ$. We have that $KT = 1$, since it is perpendicular to both NQ and MP and hence is the shortest distance between them. Let φ denote the angle $\measuredangle KNQ$. We have that $IN = r = \frac{a\sqrt{3}}{6}$ and $NQ = \frac{a\sqrt{2}}{4}$, therefore from right triangle $\triangle INQ$, $\cos\varphi = \frac{\sqrt{6}}{3}$. From this, we can compute that $\sin\varphi = \frac{\sqrt{3}}{3}$. From right triangle $\triangle NKT$, $NK = \frac{KT}{\sin\varphi} = \sqrt{3}$. But since $\triangle ABC$ is equilateral, $NK = \frac{a\sqrt{3}}{4}$, we get that $a = 4$, as desired.

23. Evaluate

$$\sum_{j=1}^{12345} \frac{1}{24691 - 2j}\left(\prod_{k=1}^{j-1} \frac{24690 - 2k}{24691 - 2k}\right).$$

Answer: 1

Solution: Consider the noodle matching problem (courtesy of Arpit): there are 12345 noodles, we start off with one noodle, and join ends together at random. The probability of a cycle of length l is

$$\frac{1}{24691 - 2l}\left(\prod_{k=1}^{l-1} \frac{246990 - 2k}{24691 - 2k}\right)$$

as the leading coefficient is the probability we close off the loop in the lth noodle and each term in the product is the probability we do not close off the loop having already k noodles in the cycle.

Then, we will always end up with a cycle of some length, so summing over the probabilities to get a cycle of length l over all possible values of l produces an answer of one.

24. Consider the sequence of integers $\{a_n\}_{n\geq 1}$ constructed in the following way. $a_1 \geq 1$, and for $n \geq 2$ we have $a_n = (a_{n-1}^2 + 337a_{n-1})$ modulo 2022. We define the period of a sequence to be the smallest integer k such that there is an integer N such that for all $n \geq N$ we have $a_{n+k} = a_n$. Determine the sum of all possible periods of a_n.

Answer: 12

Solution: We will break our problem down using the Chinese Remainder Theorem. Taking our expression in mod 2 and 3, we find that our recursion is $a_n \equiv a_{n-1}^2 + a_{n-1} \pmod{2, 3}$, which is easily seen to have a period 1 in both residues.

Therefore, our period depends only on the sequence in mod 337. Taking our expression mod 337 we have that $a_n \equiv a_{n-1}^2 \pmod{337}$. Let k be the period of a_n. Then there is sufficiently large d such that $a_d \equiv a_{d+k} \pmod{337}$. In other words $a_d \equiv a_d^{2^k} \implies \text{ord}_{337}(a_d)|2^k - 1 \pmod{337}$. But we also have that $\text{ord}(a_d)|\phi(337) = 336 = 2^4 * 3 * 7$, which implies that $\text{ord}(a_d) \in \{1, 3, 7, 21\}$. Taking the minimum k that satisfy these conditions we have that we can have $k = 1, 2, 3, 6$. We need only show that these are all possible periods.

Clearly a period of length 1 is possible since we could have $a_1 = 1$. To obtain a period of length 2 we note that taking $a_1 = 128$ gives the sequence $128, 208, 128, \ldots$. To obtain a period of length 3, observe that taking $a_1 = 8$ gives the sequence $8, 64, 52, 8, \ldots$. To obtain a period of length 6, note that taking $a_1 = 2$ gives the sequence $2, 4, 16, 256, 158, 26, 2, \ldots$.

Hence our answer is $\boxed{12}$.

25. Suppose that a, b, c are real numbers which satisfy $a^2 + b^2 + c^2 = 2022$. Let $x = \sqrt{2022 - c^2}$ and $y = \sqrt{2022 - 2ac}$. Find the minimum value of

$$\frac{xy \cdot (x + y + c)}{b^2 c}.$$

Answer: 4

Solution: Let $\triangle ABC$ be a triangle with vertices at $A = (0, 0)$, $B = (c, 0)$, and $C = (a, b)$. It is clear that the area of $\triangle ABC$ is given by $\frac{1}{2}bc$. Furthermore, we have $AB = c$, $BC = \sqrt{(a - c)^2 + b^2} = \sqrt{2022 - ac} = y$, and $AC = \sqrt{a^2 + b^2} = \sqrt{2022 - c^2} = x$.

Let R and r be the circumradius and inradius of $\triangle ABC$ respectively. Then, we have that

$$R = \frac{xyc}{4 \cdot bc/2} = \frac{xy}{2b},$$

and

$$r = \frac{bc}{x + y + c}.$$

Thus, by Euler's inequality, we can see that

$$\frac{R}{r} = \frac{xy \cdot (x + y + c)}{2b^2 c} \geq 2.$$

Thus, multiplying both sides by 2, we obtain

$$\frac{xy \cdot (x + y + c)}{b^2 c} \geq 4,$$

where equality holds when $\triangle ABC$ is equilateral. This occurs when $a = \frac{\sqrt{1011}}{2}$, $b = \frac{3\sqrt{337}}{2}$, $c = \sqrt{1011}$.