1. David flips a fair coin five times. Compute the probability that the fourth coin flip is the first coin flip that lands heads.

   Answer: $\mathbf{\dfrac{1}{16}}$

   Solution: David must flip three tails, then heads. This happens with probability $\left(\dfrac{1}{2}\right)^4 = \boxed{\dfrac{1}{16}}$.

2. Find the largest integer that divides $p^2 - 1$ for all primes $p > 3$.

   Answer: 24

   Solution: For $p = 5$, $p^2 - 1 = 24$ and hence the answer must be a divisor of 24. Since $p^2 - 1 = (p+1)(p-1)$ and all primes $p > 3$ are odd, one of $(p+1)$ and $(p-1)$ will be divisible by 2 and the other will be divisible by 4. So the answer is either 8 or 24. Furthermore, since $p > 3$ cannot be divisible by 3, either $(p+1)$ or $(p-1)$ must be divisible by 3. Hence, the answer is $\boxed{24}$.

3. We say that a number is *arithmetically sequenced* if the digits, in order, form an arithmetic sequence. Compute the number of 4-digit positive integers which are arithmetically sequenced.

   Answer: 30

   Solution: There are 9 numbers with an arithmetic sequence of difference 0 (1111 through 9999). There are 6 with an arithmetic sequence of difference 1 (1234 through 6789). There are 3 with an arithmetic sequence of difference 2 (1357 through 3579). There are 7 with an arithmetic sequence of difference -1 (3210 through 9876). There are 4 with an arithmetic sequence of difference -2 (6420 through 9753), and there is 1 with a difference of -3 (9630). The answer is therefore $9 + 6 + 3 + 7 + 4 + 1 = \boxed{30}$.

4. For any positive integer $x \geq 2$, define $f(x)$ to be the product of the distinct prime factors of x. For example, $f(12) = 2 \cdot 3 = 6$. Compute the number of integers $2 \leq x < 100$ such that $f(x) < 10$.

   Answer: 23

   Solution: Clearly, $f(x) \leq x$, so we can start with $2, 3, 4, 5, 6, 7, 8, 9$. Then, any other x with $f(x) < 10$ will be one of these numbers multiplied by a prime (or multiple primes) that already divides it. Otherwise, if we multiply by a new prime, then that prime will contribute to $f(x)$ and make it ≥ 10.

   Hence, we can investigate each of our starting numbers in turn. The new numbers in each row are bolded:

   $2 \to 2^n \to 4, 8, \mathbf{16}, \mathbf{32}, \mathbf{64}$

   $3 \to 3^n \to 9, \mathbf{27}, \mathbf{81}$

   $4 \to 2^n \to 8, 16, 32, 64$

   $5 \to 5^n \to \mathbf{25}$

   $6 \to 2^m \cdot 3^n \to \mathbf{12}, \mathbf{18}, \mathbf{24}, \mathbf{36}, \mathbf{48}, \mathbf{54}, \mathbf{72}, \mathbf{96}$

   $7 \to 7^n \to \mathbf{49}$

   $8 \to 2^n \to 16, 32, 64$

   $9 \to 3^n \to 27, 81$

Hence, we have $2, 3, 4, 5, 6, 7, 8, 9, 12, 16, 18, 24, 25, 27, 32, 36, 48, 49, 54, 64, 72, 81, 96$, for a total of $\boxed{23}$ numbers that work.

5. Compute the number of ways there are to select three distinct lattice points in three-dimensional space such that the three points are collinear and no point has a coordinate with absolute value exceeding 1.

   Answer: 49

   Solution: Each dimension can be considered independently. There are five valid arrangements for the points of each dimension: $(-1, -1, -1)$, $(0, 0, 0)$, $(1, 1, 1)$, $(-1, 0, 1)$, and $(1, 0, -1)$. Naively, this gives us $5^3 = 125$ different arrangements, but note that this counts all 27 arrangements where the points are not distinct. Therefore, we have $125 - 27 = 98$ arrangements. Since order doesn't matter, this method double counts everything, so our final answer is $98/2 = \boxed{49}$.

6. Fred lives on one of 10 islands sitting in a vast lake. One day a package drops uniformly at random on one of the ten islands. Fred can't swim and has no boat, but luckily there is a teleporter on each island. Each teleporter teleports to only one of the other ten teleporters (note that it may teleport to *itself*), and no two teleporters teleport to the same teleporter. If the configuration of the teleporters is chosen uniformly at random from all configurations that satisfy these constraints, compute the probability that Fred can get to the package using the teleporters.

   Answer: $\mathbf{\dfrac{11}{20}}$

   Solution 1: Let a_n denote the probability that Fred can get the package if there are n islands. With probability $1/n$, the package lands on his own island, so he immediately succeeds. Otherwise, his only option is to take the teleporter. With probability $1/n$, this lands him back on his current island, and he is stuck. Otherwise, he teleports to a new island. We notice that of the remaining $n-1$ teleporters with unknown destination, exactly one goes to an island already visited (namely, Fred's starting island). So, this case is identical to the case of the original problem with $n-1$ islands (alternatively, imagine combining Fred's starting island and the one to which he teleports into one island, and note that the remaining $n-1$ teleporters now all point to distinct islands). This gives us the recurrence

$$a_n = \frac{1}{n} + \frac{n-1}{n} a_{n-1}.$$

   By inspection, $a_2 = 1$. By induction, it can be shown that

$$a_n = \frac{1}{2n} + \frac{1}{2},$$

   so the answer is $\boxed{\dfrac{11}{20}}$.

   Solution 2: Each possible configuration of the island teleporters can be viewed as the permutation p of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ with island 1 teleporting to p_1, island 2 teleporting to p_2... First, the package may land on Fred's island, in which case he doesn't need to use any teleporters. This has probability $\frac{1}{10}$.

   With probability $\frac{9}{10}$ the package lands on a different island. Then we proceed by induction on the number of islands n. Assume WLOG that Fred is on island 1 and the package is on island n. For $n = 2$, Fred can only find the package for permutation $\{2, 1\}$ and not $\{1, 2\}$, and hence the probability of success is $1/2$. Assume it is true for $n-1$ islands, and consider the case of n islands. Fred can only succeed if $p_1 \neq 1$. If $p_1 = n$ (probability $1/n$) then success is guaranteed. For the $n-2$ cases $2 \leq p_1 \leq n-1$, we reduce back to our inductive hypothesis and find a success rate of $1/2$. Hence, Fred's chance of success is $\frac{1}{2}$.

Hence, when $n = 10$, fred's overall chance of success is $\frac{1}{10} + \frac{9}{10} \cdot \frac{1}{2} = \boxed{\frac{11}{20}}$

7. A one-player card game is played by placing 13 cards (Ace through King in that order) in a circle. Initially all the cards are face-up and the objective of the game is to flip them face-down. However, a card can only be flipped face-down if another card that is '3 cards away' is face-up. For example, one can only flip the Queen face-down if either the 9 or the 2 (or both) are face-up.

   A player wins the game if they can flip all but one of the cards face-down. Given that the cards are distinguishable, compute the number of ways it is possible to win the game.

   Answer: $\mathbf{13(2)^{11} = 26624}$

   Solution: We can think of an equivalent simpler game in which the cards are arranged in a circle in the following order: A, 4, 7, 10, K, 3, etc. In this game one can flip a card face-down if an adjacent card is face-up.
   In our equivalent game, we can call a face-up card whose two adjacent cards are face-down (and thus cannot itself be flipped facedown) an **isolated card**. Winning a game thus amounts to creating one and only one isolated card.
   It is clear that an isolated card cannot be created on the first turn. Moreover, if one creates an isolated card on the second turn, then one will have to create at least one more isolated card and thus the game cannot be won as such. Similarly, the game cannot be won if an isolated card is created on the 3^{rd}, 4^{th}, 5^{th}, ..., 11^{th} turn. Hence the isolated card must be created on the last, or 12^{th}, turn.
   Therefore, since the cards are distinguishable, one has 13 choices for the first card and 2 for any subsequent card. This is because any subsequent card must be one chosen from the two cards on either side of the 'face-down block' (otherwise more than one isolated card is created). Therefore the number of ways one can win is $13(2)^{11} = \boxed{26624}$.

8. $a_0, a_1, \dots$ is a sequence of positive integers where $a_n = n!$ for all $n \leq 3$. Moreover, for all $n \geq 4$, a_n is the smallest positive integer such that

   $$\frac{a_n}{a_i a_{n-i}}$$

   is an integer for all integers i, $0 \leq i \leq n$. Find a_{2014}.

   Answer: $\mathbf{2^{1007}3^{671}}$

   Solution: Note that it is equivalent to saying that for $n \geq 4$, a_n is the least common multiple of

   $$\{a_1 a_{n-1}, a_2 a_{n-2}, \dots, a_{n-1} a_1\}.$$

   This formulation makes it clear that a_n has no prime factors other than 2 and 3, for any n. We also notice that we can consider these two prime factors independently, since the LCM of the above set will be the highest power of two present times the highest power of three present.

   In particular, let b_n denote the sequence defined as above, except $b_0 = b_1 = 1$ and $b_2 = b_3 = 2$. Similarly, let c_n denote the sequence defined as above, except $c_0 = c_1 = c_2 = 1$ and $c_3 = 3$. It is clear that $a_n = b_n \cdot c_n$ for all n.

We claim that $b_n = 2^{\lfloor n/2 \rfloor}$. The proof is by induction. If $n = 2k$, then

$$b_i \cdot b_{2k-i} = 2^{\lfloor i/2 \rfloor} \cdot 2^{\lfloor (2k-i)/2 \rfloor} \le 2^{i/2} \cdot 2^{k-i/2} = 2^k,$$

with equality when $i \equiv 0 \pmod 2$. And, if $n = 2k+1$, then

$$b_i \cdot b_{2k+1-i} = 2^{\lfloor i/2 \rfloor} \cdot 2^{\lfloor (2k+1-i)/2 \rfloor} = 2^{i/2} \cdot 2^{(2k+1-i)/2} \cdot 2^{-1/2} = 2^k,$$

because exactly one of i and $2k+1-i$ is odd.

A similar argument, with 3 cases modulo 3, works to show $c_n = 3^{\lfloor n/3 \rfloor}$. Hence, in general we have $a_n = 2^{\lfloor n/2 \rfloor} 3^{\lfloor n/3 \rfloor}$, and the answer follows as $\boxed{2^{1007}3^{671}}$.

9. Compute the smallest positive integer n such that the leftmost digit of 2^n (in base 10) is 9.

Answer: 53

Solution: First, we note that for $n \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ we get the leftmost digits

$\{1, 2, 3, 4, 5, 6, 8\}$. This tells us that $n \ge 10$ or in other words, $n = 10m + r$ for some integer $m \ge 1$ and $0 \le r < 10$. But we also note that $2^{10} = 1024$ is just slightly greater than $1000 = 10^3$, so for small values of m, the leftmost digit of 2^n is the same as the leftmost digit of 2^r. But for r between 0 and 9, there is no leftmost digit 9. The closest we get is 8, so we guess that the first natural number n so that 2^n has leftmost digit 9 has $n = 10m + 3$. To finish the problem, we need to figure out the smallest integer m so that $10 > 1.024^m \cdot 8 \ge 9$ and the answer will follow. We approximate $1.024^m = (1 + .024)^m \gtrsim 1 + .024m + .024^2 \cdot \frac{m(m-1)}{2}$. And for $m = 5$ we get $1.024^5 \cdot 8 \gtrsim (1 + 5 \cdot .024 + 10 \cdot .024^2) \cdot 8 = (1.12576) \cdot 8 > 1.125 \cdot 8 = 9$. So $m = 5$ is big enough and 53 should already be our top guess. However, we do not know yet if $1.024^5 \cdot 8 < 10$ and if $m = 4$ works. First we show that $m = 4$ doesn't work: $1.024^4 \cdot 8 = 1.048576^2 \cdot 8 < 1.05^2 \cdot 8 = 1.1025 \cdot 8 < 1.125 \cdot 8 = 9$. Thus, $m > 4$. Now we check that $1.024^5 \cdot 8 < 10$. If this is true, then we are done. Using the calculations above, $1.024^5 \cdot 8 < 1.1025 \cdot 1.024 \cdot 8 < 1.11 \cdot 1.03 \cdot 8 = 1.1433 \cdot 8 < 1.25 \cdot 8 = 10$. Thus, $m > 4$ and $m = 5$ works, giving us the integer $n = 5 \cdot 10 + 3 = \boxed{53}$ is the smallest integer n with the desired property.

One may criticize that this problem has too much computation. However, the most complex computations in this problem are easily simplified: for example, $1.024^2 = 1 + 2 \cdot .024 + .024^2 = 1.048576$ is trivial (I wrote this solution down without a calculator and did all the computations with little hesitation). Also, because all computations are for strict inequalities, there are plenty of ways of minimizing the actual need to calculate things by hand. Finally, the smartest won't go about proving all of the things I proved and would stop at $1.024^m \cdot 8 \approx 8 + 8 \cdot .024m = 8 + .192m$ is really really close to 9 when $m = 5$, and since that approximation is an underestimate, one concludes intuitively that $1.024^5 \cdot 8$ is greater than 9, but only by a little bit- making $m = 5$ the most suitable choice, and the answer $\boxed{53}$.

10. Compute the number of permutations of $1, 2, 3, \ldots, 50$ such that if m divides n the mth number in the permutation divides the nth number.

Answer: 4320

Solution: Let $p_1 \ldots p_{50}$ denote a permutation of $1 \ldots 50$. Then the constraint in the problem is $m|n \Rightarrow p_m|p_n$. Of course, $p_m|p_n$ for exactly as many m, n as $m|n$, so in fact $m|n$ if and only if $p_m|p_n$.

Hence, for all n, p_n must have the same number of factors as n, and the same number of multiples as n up to 50 (which means $\lfloor \frac{50}{n} \rfloor = \lfloor \frac{50}{p_n} \rfloor$). We can construct equivalence sets of numbers that have the same number of factors and multiples. The only sets with more than one element are: $\{14, 15\}$, $\{18, 20\}$, $\{21, 22\}$, $\{17, 19, 23\}$, $\{28, 32, 44, 45, 50\}$, $\{30, 40, 42\}$, $\{26, 27, 33, 34, 35, 38, 39, 46\}$, and $\{29, 31, 37, 41, 43, 47\}$.

Clearly, for a permutation to satisfy the constraint of the problem we can only permute numbers within these equivalence sets. Additionally, we can only permute numbers that don't have unpermutable proper divisors (other than divisors shared with all elements of the set), otherwise permuting that number but not its divisor will also violate the condition in the problem. Hence, we're left with $\{17, 19, 23\}$, $\{34, 38, 46\}$, and $\{29, 31, 37, 41, 43, 47\}$. There are 3! permutations of the first set, which will uniquely determine the order of the numbers in the second set. The remaining six primes in the third set can then be in any order, so there are a total of $3! \cdot 6! = \boxed{4320}$ permutations.