1. Let $f_1(n)$ be the number of divisors that n has, and define $f_k(n) = f_1(f_{k-1}(n))$. Compute the smallest integer k such that $f_k(2013^{2013}) = 2$.

   Answer: 4

   Solution: We know that $2013^{2013} = 3^{2013} \cdot 11^{2013} \cdot 61^{2013}$. Therefore, $f_1(2013^{2013}) = 2014^3$. $2014 = 2 \cdot 19 \cdot 53$, so 2014^3 has $4^3 = 64$ divisors. $f_1(64) = 7$, and $f_1(7) = 2$. This means that $f_4(2013^{2013}) = 2$, so $k = \boxed{4}$.

2. In unit square $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E. Let M be the midpoint of $\overline{CD}$, with $\overline{AM}$ intersecting $\overline{BD}$ at F and $\overline{BM}$ intersecting $\overline{AC}$ at G. Find the area of quadrilateral $MFEG$.

   Answer: $\mathbf{\frac{1}{12}}$

   Solution: Let (ABC) denote the area of polygon ABC. Note that $\triangle AFB \sim \triangle MFD$ with $AB/MD = 2$, so we have $DF = \frac{1}{3}BD$. This implies that $(MFD) = \frac{1}{3}(MBD) = \frac{1}{3}(\frac{1}{2}(CBD)) = \frac{1}{12}$. By symmetry, $(MGC) = \frac{1}{12}$ as well. Therefore, we have $(MFEG) = (CED) - (MBD) - (MGC) = \frac{1}{4} - \frac{1}{12} - \frac{1}{12} = \boxed{\dfrac{1}{12}}$.

3. Nine people are practicing the triangle dance, which is a dance that requires a group of three people. During each round of practice, the nine people split off into three groups of three people each, and each group practices independently. Two rounds of practice are different if there exists some person who does not dance with the same pair in both rounds. How many different rounds of practice can take place?

   Answer: 280

   Solution 1: Given a permutation of nine people, let us have the first three people be in one group, the second three people in another group, and the last three people in a third group. We want to compute how many permutations generate the same group. Note that there are $(3!)^3$ ways to permute people within each group, and there are 3! ways to permute the overall groups, so the answer is $\dfrac{9!}{(3!)^4} = \boxed{280}$.

   Solution 2: Note that if three people are doing this, there is trivially exactly one unique iteration.

   If six people are doing this, then arbitrarily label one person. There are $\binom{5}{2}$ groups that can be created with this person, and then the other three people are forced to be in a group, so there are $\binom{5}{2}$ iterations for six people.

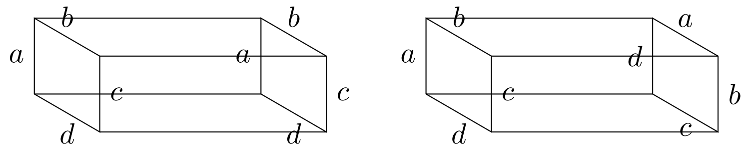
   If nine people are doing this, then arbitrarily label one person. There are $\binom{8}{2}$ groups that can be created with this person, and then the other six people can form groups in $\binom{5}{2}$ ways, so there are $\binom{5}{2} \cdot \binom{8}{2} = \boxed{280}$ iterations for nine people.

4. For some positive integers a and b, $(x^a + abx^{a-1} + 13)^b(x^3 + 3bx^2 + 37)^a = x^{42} + 126x^{41} + \cdots$. Find the ordered pair (a, b).

   Answer: $\mathbf{(7, 3)}$

   Solution: The first term is just $(x^a)^b(x^3)^a = x^{ab+3a}$, so $ab + 3a = 42$. Using the binomial theorem, the second term is $(\binom{b}{1}(x^a)^{b-1}(abx^{a-1}))(x^3)^a + (\binom{a}{1}(x^3)^{a-1}(3bx^2))(x^a)^b = (ab^2 + 3ab)x^{ab+3a-1}$, so $ab^2 + 3ab = 126$. Factoring these two equations gives $a(b + 3) = 42$ and $ab(b+3) = 126$. Dividing the second equation by the first gives $b = \boxed{3}$. Then, substituting that into the first equation gives $a = \boxed{7}$.

5. A polygonal prism is made from a flexible material such that the two bases are regular 2^n-gons ($n > 1$) of the same size. The prism is bent to join the two bases together without twisting, giving a figure with 2^n faces. The prism is then repeatedly twisted so that each edge of one base becomes aligned with each edge of the other base exactly once. For example, when $n = 2$, the untwisted and one-twist cases are shown below; in both diagrams, each edge of one base is to be aligned with the edge of the other base with the same label.



$n = 2$, untwisted $n = 2$, one twist

For an arbitrary n, what is the sum of the number of faces over all of these configurations (including the non-twisted case)?

Answer: $\boldsymbol{(n+2)2^{n-1}}$

Solution: There are 2^n cases, which can be considered based on their divisibility by powers of 2. Suppose we twist the prism by $\frac{x}{2^n}$ of a full rotation where $x = 2^k \cdot y$ and y is odd. Note that under this twist, each of the original 2^n faces is linked to every x^{th} face. Since y is odd, we see that the first multiple of x divisible by 2^n is $2^{n-k} \cdot x$. Thus, each face in the new figure is made up of 2^{n-k} faces from the original (untwisted) form and there are $\frac{2^n}{2^{n-k}} = 2^k$ sides for this figure. Next, we must consider how many rotations have x of the form $2^k \cdot y$ for a fixed value of k. The number of integers less than or equal to 2^n that are divisible by 2^k and not 2^{k+1} is $\frac{2^n}{2^k} - \frac{2^n}{2^{k+1}}$. For $k < n$, this equals 2^{n-k-1}. Finally, we add in the untwisted case, with 2^n faces. Thus, the total number of sides is $\sum_{k=0}^{n-1} 2^k \cdot 2^{n-k-1} + 2^n = n2^{n-1} + 2^n = \boxed{(n+2)2^{n-1}}$.

6. How many distinct sets of 5 distinct positive integers A satisfy the property that for any positive integer $x \le 29$, a subset of A sums to x?

Answer: 4

Solution: Let $a_1, \ldots, a_5$ denote the 5 elements of A in increasing order. Let S denote $\sum_{i=1}^{5} a_i$.

First, note that $1, 2 \in A$ because there are no other ways to obtain 1 and 2. Hence, we only have to think about the other three elements of A.

Note that there are $2^5 - 1 = 31$ non-empty subsets of A, so at most two of those subsets are not useful to the subset-sum constraint, either by having sum greater than 29 or being redundant with another subset.

We condition on the value of S. This sum is clearly at least 29, and must be at most 31, since S, $S - 1$, and $S - 2$ are all achievable subset-sums, so we require $S - 2 \le 29$.

If $S = 31$, then $A \setminus \{1\}$ has sum 30, so each subset has a distinct sum. Therefore, $4 \in A$ because the only other way to get 4 would be $1 + 3$, but $3 \in A$ would imply there were two different ways to get 3, namely $1 + 2$ and 3. Similarly, since the subsets of $\{1, 2, 4\}$ can sum to any positive integer less than 8, $8 \in A$. $16 \in A$ for the same reason for the set $\{1, 2, 4, 8\}$, and so A is completely determined.

If $S = 30$, we may have exactly one pair of subsets with the same sum. Hence, we still get $4 \in A$ because $3 \in A$ would imply $a_1 + a_2 = a_3$ and $a_1 + a_2 + a_5 = a_3 + a_5$. Similarly, $8 \in A$. Finally, since we know all elements of A must sum to 30, we choose $a_5 = 15$.

If $S = 29$, then we still have $4 \in A$ because there will be more than two redundant pairs of subsets if $3 \in S$. In general, we cannot have $x + y = z$ for $x, y, z \in A$ because there would be too many redundant sets. Hence, $a_4 \geq 7$. It can be at most 8, since otherwise there would be no way to achieve a sum of 8, so there are two cases for a_4. Each choice of a_4 determines a_5 by the condition on S. We can verify that $a_4 = 7, a_5 = 15$ works because $\{1, 2, 4\}$ can generate all sums ≤ 7, so $\{1, 2, 4, 7\}$ can generate all sums ≤ 14. Adding 15 clearly yields all sums ≤ 29. The other case can be checked trivially.

Hence, in total there are $\boxed{4}$ viable sets.

7. Find all real values of u such that the curves $y = x^2 + u$ and $y = \sqrt{x - u}$ intersect in exactly one point.

Answer: $\mathbf{(-1, 0) \cup \frac{1}{4}}$

Solution: There are two possibilities: either the curves $y = x^2 + u$ and $x = y^2 + u$ intersect in exactly one point, or they intersect in two points but one of the points occurs on the branch $y = -\sqrt{x - u}$.

Case 1: the two curves are symmetric about $y = x$, so they must touch that line at exactly one point and not cross it. Therefore, $x = x^2 + u$, so $x^2 - x + u = 0$. This has exactly one solution if the discriminant, $(-1)^2 + 4(1)(u) = 1 + 4u$, equals 0, so $u = \boxed{\frac{1}{4}}$.

Case 2: $y = x^2 + u$ intersects the x-axis at $\pm\sqrt{-u}$, while $y = \sqrt{x - u}$ starts at $x = u$ and goes up from there. In order for these to intersect in exactly one point, we must have $-\sqrt{-u} < u$, or $-u > u^2$ (note that $-u$ must be positive in order for any intersection points of $y = x^2 + u$ and $x = y^2 + u$ to occur outside the first quadrant). Hence we have $u(u + 1) < 0$, or $u \in (-1, 0)$.

8. Rational Man and Irrational Man both buy new cars, and they decide to drive around two racetracks from time $t = 0$ to time $t = \infty$. Rational Man drives along the path parametrized by

$$x = \cos(t)$$
$$y = \sin(t)$$

and Irrational Man drives along the path parametrized by

$$x = 1 + 4\cos\frac{t}{\sqrt{2}}$$
$$y = 2\sin\frac{t}{\sqrt{2}}.$$

Find the largest real number d such that at any time t, the distance between Rational Man and Irrational Man is not less than d.

Answer: $\mathbf{\frac{\sqrt{33}-3}{3}}$

Solution: We can un-parametrize this equations easily to see that Rational Man is traveling along the circle

$$x^2 + y^2 = 1$$

with a period of 2π, while Irrational Man is travelling along the ellipse

$$\frac{(x-1)^2}{16} + \frac{y^2}{4} = 1$$

with a period of $2\pi\sqrt{2}$.

Now, we claim that d is equal to the smallest distance between a point on the given circle and a point on the given ellipse. This is because for any number $r \in [0, 1)$, we can find a positive integer multiple of $\sqrt{2}$ whose fractional part is arbitrarily close to r, using a Pigeonhole argument. More precisely, for any $n \in \mathbb{N}$, we consider $\sqrt{2}, 2\sqrt{2}, \ldots, n\sqrt{2}$. Now divide the region between 0 and 1 into n equally-spaced intervals. For a given $r \in [0, 1)$, find the interval it falls into. Either one of our n multiples of $\sqrt{2}$ falls into this interval (and thus is at most $\frac{1}{n}$ from r), or none of them do, in which case two numbers fall into the same interval, and thus their difference has fractional part of magnitude less than $\frac{1}{n}$. Now, it is clear that we can take a multiple of this number that is within $\frac{1}{n}$ of r. There is a slight complication if this number is negative, but we simply approximate $1 - r$ instead of r and then multiply by -1 to get a positive number with fractional part close to r.

Applying this fact to our problem, we consider Rational Man's position at any time t. This is the same as his position at time $t + 2\pi n$ for all $n \in \mathbb{N}$. Now, if Irrational Man assumes some position at time t', then he also assumes it at time $t' + 2\pi m\sqrt{2}$ for all $m \in \mathbb{N}$. By the fact proven above, we can always choose an m such that $t' + 2\pi m\sqrt{2}$ is arbitrarily close to $t + 2\pi n$ for some $n \in \mathbb{N}$ (divide through by 2π to make this clearer). Since the two drivers can get arbitrarily close to any pair of points on their respective paths, d must simply be the shortest distance between these two paths.

Now we make the observation that given a circle of radius r centered at O and a point P outside this circle, the shortest distance from P to the circle is along the line that passes through O. This is evident by applying the Triangle Inequality to triangle OPQ, where Q is any point on the circle that is not on the line OP. Hence, minimizing distance between the ellipse and the circle is equivalent to minimizing distance between the ellipse and the center of the circle, i.e. the origin.

Hence, we set out to minimize

$$x^2 + y^2$$

subject to the constraint

$$\frac{(x-1)^2}{16} + \frac{y^2}{4} = 1,$$

so we are minimizing

$$x^2 + 4 - \frac{1}{4}(x-1)^2 = \frac{1}{4}(3x^2 + 2x + 15).$$

This attains its minimum value at $x = -\frac{1}{3}$, so the minimum squared distance from the origin is $\frac{11}{3}$. We want one less than the distance to the origin as our final answer, so report $\boxed{\dfrac{\sqrt{33}-3}{3}}$.

9. Charles is playing a variant of Sudoku. To each lattice point (x, y) where $1 \le x, y < 100$, he assigns an integer between 1 and 100, inclusive. These integers satisfy the property that in any row where $y = k$, the 99 values are distinct and are never equal to k; similarly for any column where $x = k$. Now, Charles randomly selects one of his lattice points with probability

proportional to the integer value he assigned to it. Compute the expected value of $x+y$ for the chosen point (x, y).

Answer: $\frac{\mathbf{14951}}{\mathbf{150}}$

Solution: We claim that when 100 is replaced by n, the answer is $n - \frac{n-2}{3n} = \frac{3n^2-n+2}{3n}$.

By symmetry and linearity of expectation, we need only compute the expected value of y, then multiply by two.

First, each $i = 1, \ldots, n-1$ is seen $n-2$ times (once in each row except for row i), while n is present in every row. Hence, the sum of all values is

$$\begin{aligned} n(n-1) + (n-2)\sum_{i=1}^{n-1} i &= n(n-1) + \frac{1}{2}n(n-1)(n-2) \\ &= n(n-1)\left(1 + \frac{1}{2}(n-2)\right) \\ &= \frac{1}{2}n^2(n-1). \end{aligned}$$

Meanwhile, the sum of values in row i has weight

$$\frac{1}{2}n(n+1) - i = \frac{n(n+1)-2i}{2}.$$

Hence, the desired expectation is

$$\begin{aligned} \sum_{i=1}^{n-1} \frac{(n(n+1)-2i)}{n^2(n-1)} i &= \frac{n+1}{n(n-1)}\sum_{i=1}^{n-1} i - \frac{2}{n^2(n-1)}\sum_{i=1}^{n-1} i^2 \\ &= \frac{n+1}{2} - \frac{2n-1}{3n} \\ &= \frac{3n^2-n+2}{6n}. \end{aligned}$$

Multiplying by two gives the final answer.

10. A unit circle is centered at the origin and a tangent line to the circle is constructed in the first quadrant such that it makes an angle $5\pi/6$ with the $y-$axis. A series of circles centered on the $x-$axis are constructed such that each circle is both tangent to the previous circle and the original tangent line. Find the total area of the series of circles.

Answer: $\frac{\pi(2+\sqrt{3})^2}{8\sqrt{3}} = \frac{\pi(7+4\sqrt{3})}{8\sqrt{3}} = \pi\left(\frac{1}{2} + \frac{7}{8\sqrt{3}}\right) = \pi\left(\frac{1}{2} + \frac{7\sqrt{3}}{24}\right)$

Solution: Let $\alpha = 5\pi/6$. First, notice that because the tangent line has constant slope, the intersection point on every circle must occur at the same angle with respect to the circle's center. Likewise, the line between the intersection points on two circles must coincide with the tangent line. Let circle 1 have radius R and center at $(X, 0)$ and circle 2 have radius r. It follows that circle 2 has a center at $(X+R+r, 0)$. Thus the two points of intersection with the tangent line are $(X + R\cos\alpha, R\sin\alpha)$ and $(X+R+r+r\cos\alpha, r\sin\alpha)$. The line between these must have slope $-\cot\alpha$, so

$$\frac{r\sin\alpha - R\sin\alpha}{X+R+r+r\cos\alpha-(X+R\cos\alpha)} = -\cot\alpha \implies r = R\tan^2\alpha/2.$$

Clearly, $\alpha < \pi/2$ so $\tan^2 \alpha/2 < 1$. Thus the radii obey a geometric series:

$$\sum_{n=0}^{\infty} \pi[\tan^{2n} \alpha/2]^2 = \cos^4\left(\frac{\alpha}{2}\right) \sec \alpha.$$

Plugging in $\alpha = 5\pi/6$ gives $\boxed{\dfrac{\pi(2+\sqrt{3})^2}{8\sqrt{3}}}$.

11. What is the smallest positive integer with exactly 768 divisors? Your answer may be written in its prime factorization.

Answer: $\mathbf{73513440 = 2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}$

Solution: Note that $768 = 2^8 \cdot 3$. We can immediately upper bound the answer to $2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23$. It may be possible to increase exponents on small primes and discard larger primes to reduce the answer.

There are a few cases to consider.

(a) 5 is the largest power of 5 that divides the answer. Therefore, one of 2 and 3 must contribute the factor of 3 to the number of divisors. We have two subcases to consider at this point:
   i. 2 contributes the factor of 3. We initially set $2^2 \cdot 3$. We can destroy 19 and 23 by using 2^5 and 3^3.
   ii. 3 contributes the factor of 3. We must use 3^2, and therefore the power of 2 should be 2^7, destroying 19 and 23 also.

(b) 25 is the largest power of 5 that divides the answer. We must therefore use at least 2^3 and 3^3.

Note that the very first subcase generates the smallest product, so the answer is therefore $\boxed{2^5 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17}$.

12. Suppose Robin and Eddy walk along a circular path with radius r in the same direction. Robin makes a revolution around the circular path every 3 minutes and Eddy makes a revolution every minute. Jack stands still at a distance $R > r$ from the center of the circular path. At time $t = 0$, Robin and Eddy are at the same point on the path, and Jack, Robin, Eddy, and the center of the path are collinear. When is the next time the three people (but not necessarily the center of the path) are collinear?

Answer: $\boldsymbol{t = \frac{3}{2\pi} \arccos\left(\frac{r+\sqrt{r^2+8R^2}}{4R}\right)}$.

Solution: Define

$$\omega_1 = \frac{2\pi}{3}$$
$$\omega_2 = 2\pi.$$

Let $(x_1(t), y_1(t))$ be the location of Robin and $(x_2(t), y_2(t))$ be the location of Eddy at time t. Let the center of the path be the origin and Jack's location be $(R, 0)$. Then we have

$$x_1 = r\cos(\omega_1 t) \qquad y_1 = r\sin(\omega_1 t)$$
$$x_2 = r\cos(\omega_2 t) \qquad y_2 = r\sin(\omega_2 t).$$

The three people are collinear if and only if the slopes of the lines connecting any two people are the same, i.e.

$$\frac{y_1}{x_1 - R} = \frac{y_2}{x_2 - R}.$$

Cross multiplication and factoring gives us

$$x_1 y_2 - x_2 y_1 = R(y_2 - y_1).$$

Plugging in gives us

$$r^2 \left[\cos(\omega_1 t)\sin(\omega_2 t) - \sin(\omega_1 t)\cos(\omega_2 t)\right] = Rr\left[\sin(\omega_2 t) - \sin(\omega_1 t)\right].$$

This comes out to

$$\frac{r}{R}\sin(\omega_2 t - \omega_1 t) = \sin(\omega_2 t) - \sin(\omega_1 t).$$

Replacing $3\omega_1 = \omega_2$ (from the values given in the problem), we get

$$\frac{r}{R}\sin(2\omega_1 t) = \sin(3\omega_1 t) - \sin(\omega_1 t).$$

Using sum-to-product identities, we get

$$\frac{r}{R}\sin(2\omega_1 t) = 2\sin(\omega_1 t)\cos(2\omega_1 t).$$

Expanding the left side gives us

$$2 \cdot \frac{r}{R}\sin(\omega_1 t)\cos(\omega_1 t) = 2\sin(\omega_1 t)\cos(2\omega_1 t).$$

So either the first time the three people are collinear is when $t = \pi$ or when

$$\frac{r}{R}\cos(\omega_1 t) = \cos(2\omega_1 t).$$

Double angle identity of cosine gives us

$$\frac{r}{R}\cos(\omega_1 t) = 2\cos^2(\omega_1 t) - 1.$$

This is a quadratic in cosine. Multiplying both sides by R and solving the quadratic gives us

$$\cos(\omega_1 t) = \frac{r \pm \sqrt{r^2 + 8R^2}}{4R}.$$

The positive root is the only collinear time that occurs when Robin is still in the first quadrant. Therefore, it is the earliest time. $r < R$ implies

$$\cos(\omega_1 t) = \frac{r + \sqrt{r^2 + 8R^2}}{4R} < 1,$$

so it is in the range of the cosine function. Hence, the answer is

$$\boxed{t = \frac{3}{2\pi}\arccos\left(\frac{r + \sqrt{r^2 + 8R^2}}{4R}\right)}.$$

13. A board has 2, 4, and 6 written on it. A person repeatedly selects (not necessarily distinct) values for x, y, and z from the board, and writes down $xyz + xy + yz + zx + x + y + z$ if and only if that number is not yet on the board and is also less than or equal to 2013. This person repeats this process until no more numbers can be written. How many numbers will be written at the end of this process?

Answer: 22

Solution: We claim that N can be written on the board if and only if $N + 1$ has a prime factorization of the form $3^a5^b7^c$, where $a + b + c$ is odd. It remains to actually prove this.

Note that if we write $N = xyz + xy + yz + zx + x + y + z$, then we have that $N + 1 = (x+1)(y+1)(z+1)$. Note that the original numbers, 2, 4, and 6, are each less than the primes 3, 5, and 7, respectively. Therefore, we ensure that the only primes which can divide any valid $N + 1$ are 3, 5, and 7. Furthermore, these numbers each have exponents summing to 1, an odd integer, so therefore since we multiply three integers with an odd sum of exponents, we ensure that all numbers which remain have an odd sum of exponents.

It remains to compute all numbers of the form $3^a5^b7^c$, where each number is less than or equal to 2013 and the sum of the exponents is odd. There are $\boxed{22}$ such numbers.

14. You have a 2 meter long string. You choose a point along the string uniformly at random and make a cut. You discard the shorter section. If you still have 0.5 meters or more of string, you repeat. You stop once you have less than 0.5 meters of string. On average, how many cuts will you make before stopping?

Answer: $8 - 4 \log 2$

Solution: Let $f(x)$ be the average number of cuts you make if you start with x meters of string. For $x \in [0, \frac{1}{2})$, we have $f(x) = 0$.

To calculate $f(x)$ for $x \geq \frac{1}{2}$, say you make 1 cut that brings the length of the string to y. Then you average a total of $1 + f(y)$ cuts. Note that y is distributed uniformly at random from $\frac{x}{2}$ to x. So we average $1 + f(y)$ over $y \in [\frac{x}{2}, x]$, which gives us

$$f(x) = \frac{2}{x} \int_{x/2}^{x} f(y)\, dy + 1.$$

Now we have an initial condition and recurrence. It can be easily verified that

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, \frac{1}{2}) \\ 4x - 1, & \text{if } x \in [\frac{1}{2}, 1) \\ 5x - 2x \log x - 2, & \text{if } x \in [1, 2] \end{cases}$$

satisfies the initial condition and recurrence. Therefore the answer is $f(2) = 8 - 4 \log 2$. (Note that technically we should also prove that the above $f(x)$ is the unique solution to the recurrence. We could just look at the recurrence and see that it is completely obvious that it defines a unique function. If you would like a rigorous proof, read below where we find the solution by solving two ODEs. Sufficiently nice ODEs have unique solutions so the solution is unique.)

How did we discover this solution to the recurrence? First, let $F(x) = \int_0^x f(t)\, dt$. In terms of F, the recurrence is

$$F'(x) = \frac{2}{x}\left(F(x) - F(x/2)\right) + 1.$$

For $x \in [\frac{1}{2}, 1)$, the initial condition tells us that $F(x/2) = 0$, so the recurrence simplifies to

$$F'(x) = \frac{2}{x}F(x) + 1.$$

Also notice that we have an initial condition $F(\frac{1}{2}) = 0$. Since multiplying by $\frac{2}{x}$ is the same as differentiation for x^2, we might guess that a degree 2 polynomial solves this differential equation. If we plug in a general degree 2 polynomial, we find that we are correct and that the solution is $F(x) = 2x^2 - x$ for $x \in [\frac{1}{2}, 1)$.

Now that we know $F(x)$ on $x \in [\frac{1}{2}, 1)$, we can plug that into our original recurrence for F to get the following differential equation, valid for $x \in [1, 2]$:

$$F'(x) = \frac{2}{x}F(x) - x + 2.$$

This time we could try another degree 2 polynomial, but it won't work. Specifically, if we try out $F(x) = ax^2 + bx$ we get

$$2ax + b = (2a - 1)x + 2b + 2.$$

There is no choice of a, b that satisfies this. We need to somehow get a term involving x on one side without getting it on the other side in order to balance the x's on each side. Notice that $x^2 \log x$ will give us an x when we differentiate but not when we multiply by $\frac{2}{x}$. So that might work. And indeed it does. We can plug in $F(x) = ax^2 + bx + cx^2 \log x$, solve for the coefficients (keeping in mind the initial condition $F(1) = 1$ that we get from our previous expression for F), and get $F(x) = 3x^2 - x^2 \log x - 2x$.

Now we know $F(x)$ on all of $[0, 2]$, so we can differentiate it to get $f(x)$. The result is exactly the expression for $f(x)$ that we have above.

15. Suppose we climb a mountain that is a cone with radius 100 and height 4. We start at the bottom of the mountain (on the perimeter of the base of the cone), and our destination is the opposite side of the mountain, halfway up (height $z = 2$). Our climbing speed starts at $v_0 = 2$ but gets slower at a rate inversely proportional to the distance to the mountain top (so at height z the speed v is $(h - z)v_0/h$). Find the minimum time needed to get to the destination.

Answer: $\boldsymbol{\sqrt{2504 \log^2 2 + 2500\pi^2}}$

Solution 1: For ease of notation, let $r_0 = 100$ and $h = 4$.

Begin by flattening the cone into a sector of a circle with radius $R = \sqrt{r_0^2 + h^2}$. The problem then is equivalent to finding the optimal path from the polar point $(r, \theta) = (R, 0)$ to the point $(\frac{R}{2}, \frac{r_0}{R} \cdot \pi)$ on the flattened cone. We can find an optimal path by constructing a new "distance" metric that measures elapsed time by considering standard Euclidean distance along with a factor that accounts for velocity.

Observe that any point on the sector with "radius" (distance along the cone's surface to the center) r and height (on the cone) z satisfies

$$\frac{h - z}{h} = \frac{r}{R}$$

by similar triangles. Therefore, the speed at radius r on the sector is

$$\frac{r}{R} \cdot v_0.$$

Let the optimal path curve be given by $\gamma(\theta) = (r(\theta), \theta)$. We wish to optimize the integral that gives the total time spent along the curve γ. We can measure length by the standard polar arclength formula, and we can measure speed using the formula above. Hence, we can measure time by looking at distance divided by speed:

$$\int_{\theta=0}^{\frac{r_0}{R}\cdot\pi} \frac{\text{distance}}{\text{speed}} = \int_0^{\frac{r_0}{R}\cdot\pi} \frac{\sqrt{dr^2 + r^2\,d\theta^2}}{\frac{r}{R}\cdot v_0} = \frac{R}{v_0}\int_0^{\frac{r_0}{R}\cdot\pi} \sqrt{1 + \left(\frac{1}{r}\cdot\frac{dr}{d\theta}\right)^2}\,d\theta.$$

We now wish to find a coordinate transformation in which this path is a straight line, so that the minimum time will just be the Euclidean distance between the endpoints. We can do this by choosing a new coordinate $\tilde{r}$ so that

$$\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{d\tilde{r}}{d\theta}.$$

By integrating, it is easily seen that one such substitution is $\log r = \tilde{r}$, which results in the endpoints

$$(\log R, 0) \text{ and } \left(\log\frac{R}{2}, \frac{r_0}{R}\cdot\pi\right),$$

so our integral is

$$\sqrt{\left(\log R - \log\frac{R}{2}\right)^2 + \frac{r_0^2}{R^2}\cdot\pi^2} = \sqrt{(\log 2)^2 + \frac{r_0^2}{R^2}\cdot\pi^2}.$$

Multiplying by the constant terms we factored out of this integral earlier, our final answer (and minimum time) is

$$\sqrt{\frac{R^2}{v_0^2}(\log 2)^2 + \frac{r_0^2}{v_0^2}\cdot\pi^2} = \boxed{\sqrt{2504(\log 2)^2 + 2500\pi^2}}.$$

Solution 2: Rather than thinking of the velocity as decreasing as we climb up the mountain, we can imagine the mountain is growing larger as we climb up it. In fact, no matter how far you've climbed up the mountain, it will appear as you have made no progress. Thus, the mountain can be thought of as an infinitely tall cylinder. As a sanity check, note that at height z, the circumference of the cone will be $\frac{h-z}{h}2\pi r$, and our velocity is $\frac{h-z}{h}v_0$, so it will take $\frac{2\pi r}{v_0}$ units of time to go around the circumference. This is independent of our height z, confirming our intuition that the mountain should be treated as a cylinder.

Our problem then becomes determining where our original destination on the cone corresponds to on this cylinder. As checked above, the radius of the cylinder is the same as the radius of the base of the cone. So all that's left to determine is the height on the cylinder that corresponds to a height of $z = 2$ on the cone. We do so by evaluating the integral $\int_{\frac{s}{2}}^{s} \frac{s}{x}dx$ where s denotes the length of the slant of the cone. To justify this integral, we note that when our distance to the top of the mountain is $\frac{s}{q}$, our speed is $\frac{\frac{s}{q}}{s}$ of our original speed, so a small distance dx will take $\frac{s}{\frac{s}{q}}$ times as long to travel. This integral evaluates to $s\log 2$.

Now, travelling on a cylinder is the same as travelling on the plane, so we uncurl the cylinder. If we denote our initial position as $(0, 0)$, then our destination has coordinates $(\pi r, s\log 2)$. The distance to our destination can now be calculated by Pythagoras: $\sqrt{\pi^2 r^2 + s^2(\log 2)^2}$. The time then to get to our destination is $\frac{\sqrt{\pi^2 r^2 + s^2(\log 2)^2}}{v_0}$. We are given that $r = 100$ and $v_0 = 2$. The slant is easily calculated $s^2 = r^2 + h^2 = 10016$. Plugging these values in, we see that the time required to get to our destination is

$$\boxed{\sqrt{2504(\log 2)^2 + 2500\pi^2}}.$$